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A Comparative Study of Classical Relation and Fuzzy Relation

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Abstract

Fuzzy relation is an extension of classical relation as fuzzy set is an extension of classical set. This work represents an overview of classical relation and fuzzy relation. Some definitions related to fuzzy relation are discussed and also explained by giving example. Max-min operation and max-product operation are discussed for both classical relations and fuzzy relations. Max-avg operation is also discussed for fuzzy relations. Finally, fuzzy equivalence relation is discussed.

Keywords: Characteristic function; Membership grade; Max-min operation; Max-product operation; Max-avg operation.

1. Introduction

The concept of fuzzy set was first introduced by L. A. Zadeh in 1965. Fuzzy set is an extension or generalization of classical set [1]. A classical set is a well defined collection of objects. Each classical set can be defined by a characteristics function [5],

$$\mu_X(x) = \begin{cases} 1\,;\, x \in X \\ 0\,;\, x \notin X \end{cases}$$

The characteristics function indicates whether an element is present in the set or not. If any element is present in the set X, the membership grade is 1 and the membership grade is 0 if it is not present in the set.

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If the valuation set $\{0,1\}$ is replaced by the real interval $[0,1]$, the set is called fuzzy set and various degrees of membership are obtained. Thus, a fuzzy set is obviously a generalization of classical set.

In mathematics, a relation describes how the elements of one set, X are associated with the elements of another set, Y. If X and Y be two sets then relation between X and $Y$is a subset of the Cartesian product $X \times Y$.That is, the relation, R between X and Y can be defined by,

$$R = \{(x, y) | x \in X, y \in Y\}$$

The characteristics function of a classical relation is given by,

$$\mu_R(x, y) = \begin{cases} 1 \,;\, (x, y) \in R \\ 0 \,;\, (x, y) \notin R \end{cases}$$

Fuzzy relation is the generalization of classical relation as fuzzy set is the generalization of classical set [4]. In a fuzzy relation, membership grade $R(x, y)$ is taken from the interval [0,1] instead of $\{0, 1\}$.

2. Fuzzy Relation [4]

First we describe a classical relation between the color of a mango, X and the grade of its maturity, Y.

$$X = \{yellow, light\ green, green\ \}$$

$$Y = \{raw, half - matured, ripe\}$$

Now, the membership grade of classical relation from X to Y is given in the tabular form,

Table 1

	Raw	Half-matured	Ripe
Yellow	0	0	1
Light green	0	1	0
Green	1	0	0

We see that, the membership grade is obtained from the set, $\{0,1\}$. If the color of the mango is green then it is raw, if the color is light green then it is half-matured and if the color is yellow then the mango is ripe. Thus the characteristics function of a classical relation is given by,

$$\mu_R(x, y) = \begin{cases} 1 \,;\, (x, y) \in R \\ 0 \,;\, (x, y) \notin R \end{cases}$$

Now, if we take the membership grade from [0,1] interval instead of {0,1}, we get a membership grade in fuzzy relation. We show the membership grade in the fuzzy relation as follows:

Table 2

	Raw	Half-matured	Ripe
Yellow	0	0.8	1
Light Green	0.6	1	0.3
Green	1	0.3	0

Definition: Let, X and Y are two non-empty sets. Then the fuzzy relations, $R: X \times Y \to I$ is defined as follows:

$$R(x,y) = \{(x,y), \mu_R(x,y) | (x,y) \in X \times Y\}$$

Where, $\mu_R(x,y)$is the membership grade of the ordered pair $(x,y)$and $I = [0,1]$ [3].

Let, $X = \{x_1, x_2, \dots \dots, x_m\}$ and $Y = \{y_1, y_2, \dots \dots, y_n\}$, the $m \times n$ matrix form of the fuzzy relation is:

$$R(x,y) = \begin{bmatrix} \mu_R(x_1, y_1) & \cdots & \mu_R(x_1, y_n) \\ \vdots & \ddots & \vdots \\ \mu_R(x_m, y_1) & \cdots & \mu_R(x_m, y_n) \end{bmatrix}$$

Example: Let, $X = \{0,1,2,3,4,5\}$ and $Y = \{1,3,5,7,9,11\}$. A fuzzy relation is defined as follows:

$$R(x,y) = \begin{cases} 1 \ ; \ x \le 0 \\ 1 - \dfrac{x^2}{1+y^2} \ ; 0 < x \le y \\ 0 \ ; \ x > y \end{cases}$$

The membership grade of the relation is given in the following table:

Table 3

$\boldsymbol{R(x,y)}$	**1**	**3**	**5**	**7**	**9**	**11**
0	1	1	1	1	1	1
1	0.5	0.9	0.96	0.98	0.988	0.99
2	0	0.6	0.85	0.92	0.95	0.97
3	0	0.1	0.65	0.82	0.89	0.93
4	0	0	0.38	0.68	0.80	0.87
5	0	0	0.04	0.5	0.7	0.8

2.1. Domain, Range and Height of a Fuzzy Relation [2,5]

Let, $R: X \times Y \rightarrow I$ be a fuzzy relation. Then domain of $R(x, y)$ is defined as follows:

$$Dom(R) = \left\{\left(\max_y R(x, y)\right) \middle| (x, y) \in X \times Y\right\}$$

This is also called 1st projection or projection of $R(x, y)$ on X.

The range of $R(x, y)$ is defined as follows:

$$Range(R) = \left\{\left(\max_x R(x, y)\right) \middle| (x, y) \in X \times Y\right\}$$

This is also called 2nd projection or projection of $R(x, y)$ on Y.

And the height of $R(x, y)$ is defined as follows:

$$H(R) = \max_y \left(\max_x R(x, y)\right) |(x, y) \in X \times Y$$

This is also called global projection.

Note: If the height of a fuzzy relation, $H(R) = 1$,then the fuzzy relation is called normal fuzzy relation and if, $H(R) < 1$, the fuzzy relation is called subnormal fuzzy relation [2].

Example: Let, $X = \{1,2,3,4,5\}$ and $Y = \{3,4,5,6,7,8\}$. A fuzzy relation is defined as follows:

$$R(x, y) = \begin{cases} 1 - \dfrac{x^2}{2y^2} & ; 0 < x \leq y \\ 0 & ; \ x > y \end{cases}$$

The membership grade of the relation is given in the following table:

Table 4

$\boldsymbol{R(x, y)}$	**3**	**4**	**5**	**6**	**7**	**8**
1	0.94	0.97	0.98	0.986	0.989	0.99
2	0.78	0.875	0.92	0.94	0.96	0.97
3	0.5	0.72	0.82	0.875	0.91	0.93
4	0	0.5	0.68	0.78	0.84	0.875
5	0	0	0.5	0.65	0.74	0.80

Thus we get,

$$Dom(R) = \{0.99, 0.97, 0.93, 0.875, 0.80\}$$

$$Range(R) = \{0.94, 0.97, 0.986, 0.989, 0.99\}$$

And,

$$H(R) = 0.99$$

3. Standard Operations on relations

We will discuss three standard operations on fuzzy relations. These are max-min operation, max-product operation and max-avg operation.

3.1. Max-Min Operation [2]

Definition: If P and Q are two fuzzy relations, then max-Min operation of P and Q is defined by,

$$PoQ = {}_{y\in Y}^{max}\{min(P(x,y), Q(y,z)\}; \forall(x,y),(y,z) \in X \times Y$$

Example: If $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ be two classical relations, then max-min operations on P and Q is:

$$PoQ = \begin{bmatrix} \vee(1\wedge 1, 0\wedge 1, 1\wedge 0) & \vee(1\wedge 1, 0\wedge 0, 1\wedge 0) & \vee(1\wedge 0, 0\wedge 1, 1\wedge 1) & \vee(1\wedge 1, 0\wedge 0, 1\wedge 1) \\ \vee(1\wedge 1, 1\wedge 1, 1\wedge 0) & \vee(1\wedge 1, 1\wedge 0, 1\wedge 0) & \vee(1\wedge 0, 1\wedge 1, 1\wedge 1) & \vee(1\wedge 1, 1\wedge 0, 1\wedge 1) \\ \vee(0\wedge 1, 1\wedge 1, 1\wedge 0) & \vee(0\wedge 1, 1\wedge 0, 1\wedge 0) & \vee(0\wedge 0, 1\wedge 1, 1\wedge 1) & \vee(0\wedge 1, 1\wedge 0, 1\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} \vee(1,0,0) & \vee(1,0,0) & \vee(0,0,1) & \vee(1,0,1) \\ \vee(1,1,0) & \vee(1,0,0) & \vee(0,1,1) & \vee(1,0,1) \\ \vee(0,1,0) & \vee(0,0,0) & \vee(0,1,1) & \vee(0,0,1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Example: If $P = \begin{bmatrix} .2 & .4 & .7 \\ 0 & .6 & 1 \\ .5 & .7 & .5 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & .6 & .8 & .7 \\ .3 & .2 & 0 & .9 \\ 0 & 1 & .5 & .5 \end{bmatrix}$ be two fuzzy relations, then max-min operations on P and Q is:

$$PoQ = \begin{bmatrix} \vee(.2\wedge 1, .4\wedge .3, .7\wedge 0) & \vee(.2\wedge .6, .4\wedge .2, .7\wedge 1) & \vee(.2\wedge .8, .4\wedge 0, .7\wedge .5) & \vee(.2\wedge .7, .4\wedge .9, .7\wedge .5) \\ \vee(0\wedge 1, .6\wedge .3, 1\wedge 0) & \vee(0\wedge .6, .6\wedge .2, 1\wedge 1) & \vee(0\wedge .8, .6\wedge 0, 1\wedge .5) & \vee(0\wedge .7, .6\wedge .9, 1\wedge .5) \\ \vee(.5\wedge 1, .7\wedge .3, .5\wedge 0) & \vee(.5\wedge .6, .7\wedge .2, .5\wedge 1) & \vee(.5\wedge .8, .7\wedge 0, .5\wedge .5) & \vee(.5\wedge .7, .7\wedge .9, .5\wedge .5) \end{bmatrix}$$

$$= \begin{bmatrix} \vee(.2,.3,0) & \vee(.2,.2,.7) & \vee(.2,0,.5) & \vee(.2,.4,.5) \\ \vee(0,.3,0) & \vee(0,.2,1) & \vee(0,0,.5) & \vee(0,.6,.5) \\ \vee(.5,.3,0) & \vee(.5,.2,.5) & \vee(.5,0,.5) & \vee(.5,.7,.5) \end{bmatrix}$$

$$= \begin{bmatrix} .3 & .7 & .5 & .5 \\ .3 & 1 & .5 & .6 \\ .5 & .5 & .5 & .7 \end{bmatrix}$$

3.2. Max-Product Operation [4]

Definition: If P and Q are two fuzzy relations, then max-product operation of P and Q is defined by,

$$PoQ = \max_{y \in Y}\{(P(x,y).\,Q(y,z)\}; \forall(x,y),(y,z) \in X \times Y$$

Example: If $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ be two classical relations, then max-product operations on P and Q is:

$$PoQ = \begin{bmatrix} \vee(1.1, 0.1, 1.0) & \vee(1.1, 0.0, 1.0) & \vee(1.0, 0.1, 1.1) & \vee(1.1, 0.0, 1.1) \\ \vee(1.1, 1.1, 1.0) & \vee(1.1, 1.0, 1.0) & \vee(1.0, 1.1, 1.1) & \vee(1.1, 1.0, 1.1) \\ \vee(0.1, 1.1, 1.0) & \vee(0.1, 1.0, 1.0) & \vee(0.0, 1.1, 1.1) & \vee(0.1, 1.0, 1.1) \end{bmatrix}$$

$$= \begin{bmatrix} \vee(1,0,0) & \vee(1,0,0) & \vee(0,0,1) & \vee(1,0,1) \\ \vee(1,1,0) & \vee(1,0,0) & \vee(0,1,1) & \vee(1,0,1) \\ \vee(0,1,0) & \vee(0,0,0) & \vee(0,1,1) & \vee(0,0,1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Example: If $P = \begin{bmatrix} .2 & .4 & .7 \\ 0 & .6 & 1 \\ .5 & .7 & .5 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & .6 & .8 & .7 \\ .3 & .2 & 0 & .9 \\ 0 & 1 & .5 & .5 \end{bmatrix}$ be two fuzzy relations, then max-product operations on P and Q is:

$$PoQ = \begin{bmatrix} \vee(.2,.12,0) & \vee(.12,.08,.7) & \vee(.16,0,.35) & \vee(.14,..36,.35) \\ \vee(0,.18,0) & \vee(0,.12,1) & \vee(0,0,.5) & \vee(0,.54,.5) \\ \vee(.5,.21,0) & \vee(.3,.14,.5) & \vee(.4,0,.25) & \vee(.35,.63,.25) \end{bmatrix}$$

$$= \begin{bmatrix} .2 & .7 & .35 & .36 \\ .18 & 1 & .5 & .54 \\ .5 & .5 & .4 & .63 \end{bmatrix}$$

3.3. Max-Avg Operation [4]

Definition: If P and Q are two fuzzy relations, then max-avg operation of P and Q is defined by,

$$PoQ = \max_{y\in Y}\{\frac{1}{2}(P(x,y)+Q(y,z)\}; \forall(x,y),(y,z)\in X\times Y$$

Example: If $P=\begin{bmatrix} .2 & .4 & .7\\ 0 & .6 & 1\\ .5 & .7 & .5\end{bmatrix}$ and $Q=\begin{bmatrix} 1 & .6 & .8 & .7\\ .3 & .2 & 0 & .9\\ 0 & 1 & .5 & .5\end{bmatrix}$ be two fuzzy relations, then max-avg operations on P and Q is:

$$PoQ$$
$$=\begin{bmatrix} \vee(\frac{.2+1}{2},\frac{.4+.3}{2},\frac{.7+0}{2}) & \vee(\frac{.2+.6}{2},\frac{.4+.2}{2},\frac{.7+1}{2}) & \vee(\frac{.2+.8}{2},\frac{.4+0}{2},\frac{.7+.5}{2}) & \vee(\frac{.2+.7}{2},\frac{.4+.9}{2},\frac{.7+.5}{2})\\ \vee(\frac{0+1}{2},\frac{.6+.3}{2},\frac{1+0}{2}) & \vee(\frac{0+.6}{2},\frac{.6+.2}{2},\frac{1+1}{2}) & \vee(\frac{0+.8}{2},\frac{.6+0}{2},\frac{1+.5}{2}) & \vee(\frac{0+.7}{2},\frac{.6+.9}{2},\frac{1+.5}{2})\\ \vee(\frac{.5+1}{2},\frac{.7+.3}{2},\frac{.5+0}{2}) & \vee(\frac{.5+.6}{2},\frac{.7+.2}{2},\frac{.5+1}{2}) & \vee(\frac{.5+.8}{2},\frac{.7+0}{2},\frac{.5+.5}{2}) & \vee(\frac{.5+.7}{2},\frac{.7+.9}{2},\frac{.5+.5}{2})\end{bmatrix}$$

$$=\begin{bmatrix} \vee(.6,.35,.35) & \vee(..4,.3,..85) & \vee(.5,.2,.6) & \vee(.45,.65,.6)\\ \vee(.5,.45,.5) & \vee(.3,.4,1) & \vee(.4,.3,.75) & \vee(.35,.75,.75)\\ \vee(.75,.5,.25) & \vee(.55,.45,.75) & \vee(.65,.35,.5) & \vee(.6,.8,.5)\end{bmatrix}$$

$$=\begin{bmatrix} .6 & .85 & .6 & .65\\ .5 & 1 & .75 & .75\\ .75 & .75 & .65 & .8\end{bmatrix}$$

From the above example we see that, max-min operation and max-product operation on two classical relations give the same result. But max-min operation and max-product operation on two fuzzy relations have different result. Also the max-avg operation on two fuzzy relations has different result from max-min and max-product operation.

4. Fuzzy Equivalence relation [2]

A binary fuzzy relation R in $X\times X$ is called fuzzy equivalence relation if it satisfies the following three conditions:

i) $R(x,x)=1\,;\forall x\in X$ [Reflexive condition]

ii) $R(x,y)=R(y,x)\,;\ \forall x,y\in X$ [Symmetric condition]

iii) $R(x,z)\geq \max_{y\in Y}\{min(R(x,y),R(y,z)\}\,;\forall x,y,z\in X$ [Transitive condition]

Example: Suppose,

$$\begin{array}{cc} & \begin{matrix} x_1 & x_2 & x_3\end{matrix}\\ R=\begin{matrix} x_1\\ x_2\\ x_3\end{matrix} & \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & .6\\ 0 & .6 & 1\end{bmatrix}\end{array}$$

For reflexivity,

$$R(x_1, x_1) = 1$$

$$R(x_2, x_2) = 1$$

$$R(x_3, x_3) = 1$$

Therefore, R is reflexive.

For symmetry,

$$R(x_1, x_2) = R(x_2, x_1) = 0$$

$$R(x_1, x_3) = R(x_3, x_1) = 0$$

$$R(x_2, x_3) = R(x_3, x_2) = .6$$

Therefore, R is symmetric.

For transitivity,

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & .6 \\ 0 & .6 & 1 \end{bmatrix}$$

Now,

$$RoR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & .6 \\ 0 & .6 & 1 \end{bmatrix} \text{o} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & .6 \\ 0 & .6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \vee(1 \wedge 1, 0 \wedge 0, 0 \wedge 0) & \vee(1 \wedge 0, 0 \wedge 1, 0 \wedge .6) & \vee(1 \wedge 0, 0 \wedge 0, 0 \wedge 0) \\ \vee(0 \wedge 1, 1 \wedge 0, .6 \wedge 0) & \vee(0 \wedge 0, 1 \wedge 1, .6 \wedge .6) & \vee(0 \wedge 0, 1 \wedge .6, .6 \wedge 1) \\ \vee(0 \wedge 1, .6 \wedge 0, 1 \wedge 0) & \vee(0 \wedge 0, .6 \wedge 1, 1 \wedge .6) & \vee(0 \wedge 0, .6 \wedge .6, 1 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} \vee(1, 0, 0) & \vee(0, 0, 0) & \vee(0, 0, 0) \\ \vee(0, 0, 0) & \vee(0, 1, .6) & \vee(0, .6, .6) \\ \vee(0, 0, 0) & \vee(0, .6, .6) & \vee(0, .6, 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & .6 \\ 0 & .6 & 1 \end{bmatrix} = \text{R}$$

That is, $RoR \geq R$. So, R is transitive.

Since, the relation R satisfies all the three conditions, it is an equivalence relation.

Example: Consider another fuzzy relation, $R = \begin{bmatrix} 1 & 0 & .6 \\ 0 & 1 & .3 \\ .6 & .3 & 1 \end{bmatrix}$. The relation is not equivalence relation, because it doesn't satisfy the condition of transitivity.

5. Conclusion

Fuzzy relations generalize the concept of classical relations in the same manner as fuzzy sets generalize the concept of classical sets. Fuzzy set theory as well as fuzzy relations has significant importance in mathematics and is being used widely in many areas such as clustering, decision-making, medical diagnosis, engineering sector, stock market, even in the cricket field to predict score.

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