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Hamltonian Connectedness and Toeplitz Graphs

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Abstract

A square matrix of order n is called Toeplitz matrix if it has constant elements along all diagonals parallel to the main diagonal and a graph is called Toeplitz graph if its adjacency matrix is Toeplitz. In this paper we proved that the Toeplitz graphs $T_n\langle 2,3,t\rangle$, for $n \geq 28$ and $4 \leq t \leq \frac{n-8}{2}$ are Hamiltonian connected.

Keywords: Hamiltonian graph; Hamiltonian connected; Toeplitz graph; Toeplitz matrix; Hamiltonian path.

1. Introduction

A square matrix of order n is called **Toeplitz matrix** if it has constant elements along all diagonals parallel to the main diagonal. A simple undirected graph T_n with vertex set $\{1,2,3,\ldots,n\}$ is called a **Toeplitz graph** if its adjacency matrix is Toeplitz. A Toeplitz graph is uniquely defined by the first row of its adjacency matrix. The first row of adjacency matrix of a Toeplitz graph is always a sequence of 0's and 1's. If the 1's in that sequence places at $t_1+1, t_2+1, t_3+1, \ldots, t_k+1$ positions with $1 \leq t_1 < t_2 < t_3 < \cdots < t_k < n$, we write $T_n = T_n\langle t_1, t_2, t_3, \ldots, t_k\rangle$. In a Toeplitz graph $T_n\langle t_1, t_2, t_3, \ldots, t_k\rangle$ two vertices a and b are connected by an edge if and only if $|a-b| \in \{t_1, t_2, t_3, \ldots, t_k\}$.

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A graph G of order n is called **Hamiltonian** if it contains a cycle of order n. A graph G of order n is called **traceable** if it contains a path of order n and such a path is called **Hamiltonian path**. The graph G is called **Hamiltonian connected** if for any pair of distinct vertices a and b of G, there exists a Hamiltonian path with ends a and b. Every Hamiltonian connected graph is Hamiltonian.

Connectedness, bipartiteness, colourbality and planarity of Toeplitz graphs are discussed in [1, 2, 3, 4]. Some of the Hamiltonian properties of undirected Toeplitz graphs were discussed in [1] and [5]. The Hamiltonian properties of directed Toeplitz graphs were studied in [6, 7]. S. Malik and T. Zamfirescu investigated Hamiltonian properties of Toeplitz graphs in [8]. M. F. Nadeem Ayesha Shabbir and Tudor Zamfirescu complete the picture of Toeplitz graphs $T_n\langle t_1,t_2\rangle$, $T_n\langle 1,3,t\rangle$ and $T_n\langle 1,5,\mathrm{t}\rangle$ in [9]. In this paper we proved that the Toeplitz graphs $T_n\langle 2,3,\mathrm{t}\rangle$ are Hamiltonian connected for $n\geq 28$ and $4\leq t\leq \frac{n-8}{2}$.

Suppose T is a Toeplitz graph and l,m,k,q,n are its vertices. If $l>m$, then $P[l,l+1,m]$ is a path with ends l and $l+1$ that contains all vertices, $m,m+1,m+2,\ldots,l$. If $m>l$ then $P[l,l+1,m]$ a path with ends l and $l+1$ that contains all vertices, $l,l+1,l+2,\ldots,m$. The path $P[l,m]$ has end vertices l and m and contains all vertices, $l,l+1,l+2,\ldots,m$ except vertices $l+1$ and $m-1$. The path $H[l,m]$ has ends l and m and contains all vertices $l,l+1,l+2,\ldots,m$. The path $P'[l,m,k]$ has end vertices l and m, and contains all vertices $l,l+1,l+2,\ldots,m$ except vertex $k\in\{l+1,m-1\}$. The path $H_x^y[l,m]$ has ends x ,y *and* contains vertices $l,l+1,l+2,\ldots,m$. The path $h_x^y[l,m]$ has ends x and y contains all vertices $l,l+1,l+2,\ldots,m$ such that $l+5\leq x,y\leq m-5$.

2. Preliminaries

Lemma 2.1 The Toeplitz graph $T_n\langle 2,3\rangle$, n $\geq$ 6 admits Hamiltonian path $P[1,2,n]$ and $P[n-1,n,1]$.

Proof. The Toeplitz graph $T_{2n}\langle 2,3\rangle$ has he following Hamiltonian path $P[1,2,n]$. "$1,3,5,\ldots,2n-7,\ 2\mathrm{n}-5,$ $2\mathrm{n}-2,2\mathrm{n},2\mathrm{n}-3,2\mathrm{n}-1,2\mathrm{n}-4,2\mathrm{n}-6,\ldots 6,4,2.$"

The Toeplitz graph $T_{2n+1}\langle 2,3\rangle$ the Hamiltonian path "$1,3,5,\ldots 2n-5,2n-3,2n,2n-2,\ 2n+1,2n-1,\ 2n-4,$ $2n-6,\ldots 6,4,2$" that connects 1 and 2. Due to symmetry of the Toeplitz graphs $T_n\langle 2,3\rangle$ has Hamiltonian path $P[n-1,n,1]$. The Hamiltonian paths connecting 1 and 2 are shown in Fig. 1 and Fig. 2. □

Corollary 2.2 (a). In Toeplitz graphs $T_n\langle 2,3\rangle$, if l < m, then $P[l,l+1,\mathrm{m}]$ exists for $m-l\geq 5$.

(b). In Toeplitz graphs $T_n\langle 2,3\rangle$, if l > m, then $P[l,l+1,\mathrm{m}]$ exists for $l-m\geq 4$.

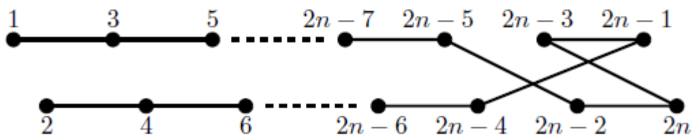

Figure 1: Hamiltonian Path with ends 1 and 2

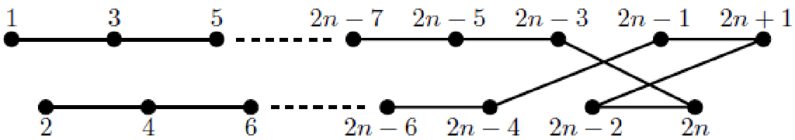

Figure 2: Hamiltonian Path with ends 1 and 2

Lemma 2.3. The Toeplitz graph $T_n\langle 2,3\rangle$ for $n \geq 8$ contains the path $P[1,n]$.

Proof. We divide the proof in following four cases.

Case 1. If $n = 0 \pmod 4$, then $n = 4k$ for $k \geq 2$. The required path is shown in Fig. 3.

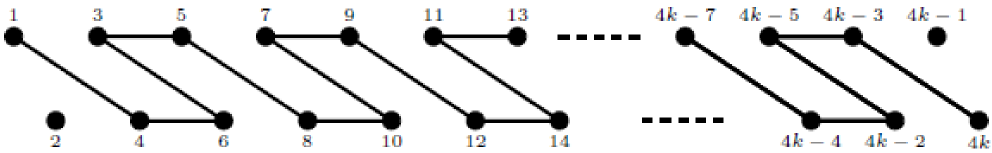

Figure 3: Path with ends 1 and 4k containing all vertices except 2 and $4k-1$

Case 2. If $n = 1 (\bmod 4)$, then n = 4k + 1 for k _ 2. The path that connects 1 with 4k + 1 and contains all the vertices in the Toeplitz graph $T_{4k+1}\langle 2,3\rangle$ except 2 and 4k is "$1,4,6,3,5,8,10,7,\ldots,4k-6,4k-9,4k-7,$ $4k-4,$ $4k-2,4k-5,4k-3,4k-1,4k+1$" and is shown in Fig. 4.

Case 3. If n = 2 (mod 4), then n = 4k + 2 for $n \geq 2$. The path that connects 1 with 4k + 2 and contains all the vertices in the Toeplitz graph $T_{4k+2}\langle 2,3\rangle$, except 2 and 4k+1 is $1,3,5,7,4,6,8,10,12,9,11,14,16,13,15,18,\ldots 4k-5$ $4k-2,4k,4k-3,4k-1,4k+2$" and is shown in Fig. 5.

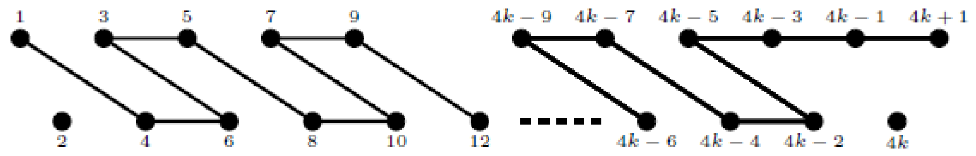

Figure 4: Path with ends 1 and 4k + 1 containing all vertices except 2 and 4k

Case 4. If n = 3 (mod 4), then n = 4k + 3 for $n \geq 2$. The path that connects 1 with 4k + 3 and contains all the vertices in the Toeplitz graph $T_{4k+3}\langle 2,3\rangle$, except 2 and 4k + 2 is "$1,3,5,7,4,6,9,11,8,10,13,\ldots 4k-5,4k-8$, $4k-6,4k-3,4k,4k-2,4k-4,4k-3,4k+1,4k+3$" and is shown in Fig. 6.

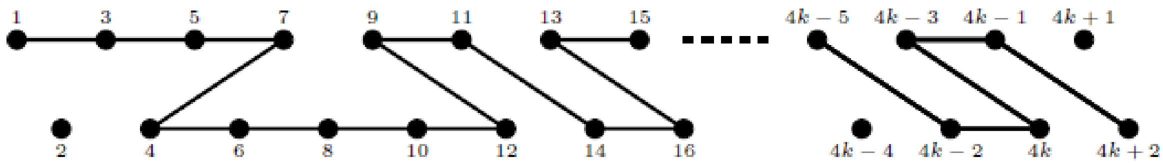

Figure 5: Path with ends 1 and 4k + 2 containing all vertices except 2 and 4k + 1

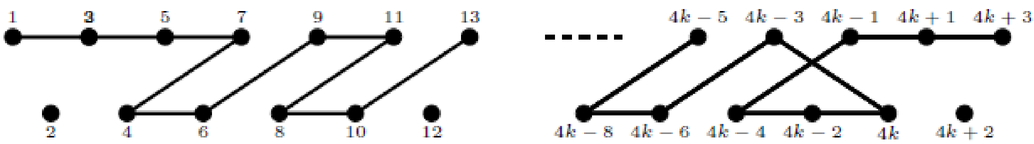

Figure 6: Path with ends 1 and 4k + 3 containing all vertices except 2 and 4k + 2

Corollary 2.4. The path $P[l,m]$ exists in Toeplitz graphs $T_n\langle 2,3\rangle$ for $m-l \geq 7$.

Lemma 2.5. The Toeplitz graph $T_n\langle 2,3\rangle$, for $n \geq 6$ and $n \neq 8$ admits a Hamiltonian path $H[1,n]$.

Proof. We will prove this result in five cases.

Case 1. If n = 1 (mod 5) then n = 5k + 1 where $k \in N$. See Fig. 7 for a Hamiltonian path from 1 to 5k + 1.

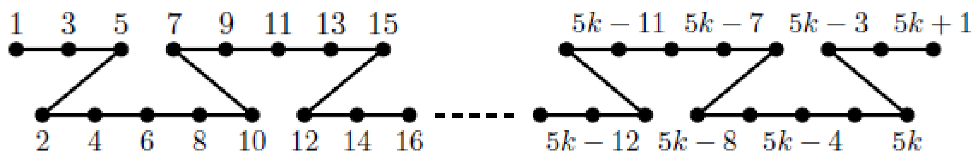

Figure 7: Hamiltonian path connecting 1 and 5k + 1 in $T_{5k+1}\langle 2,3\rangle$

Case 2. The Hamiltonian path connecting 1 and 7 in $T_7\langle 2,3\rangle$ is shown in Fig. 8. If n = 2 (mod 5), then we can write n = 5k + 2 where $k \in N$. For any Toeplitz graph $T_{5k+2}\langle 2,3\rangle$ for $k \geq 2$ joining the path of Fig. 8 from 1 to 7 with the path of Fig. 7 from 7 to 5k + 2 we get the Hamiltonian path connecting 1 and 5k + 2.

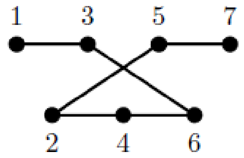

Figure 8: Hamiltonian path between 1 and 7 in $T_7\langle 2,3\rangle$

Case 3. The Hamiltonian path connecting 1 and 13 in Toeplitz graph $T_{13}\langle 2,3\rangle$, is shown in Fig. 9.

If n = 3 (mod 5), then we can write n = 5k+3 for any $k \geq 2$. The Hamiltonian path joining vertices 1 and 5k + 3 in any Toeplitz graph $T_{5k+3}\langle 2,3\rangle$ for $k \geq 3$ is obtained by joining the path of Fig. 9 with the path of Fig 7.

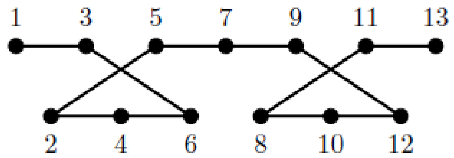

Figure 9: Hamiltonian path connecting 1 and 13 in $T_{13}\langle 2,3\rangle$

Case 4. If n = 4 (mod 5), then n = 5k + 4 for $k \in N$. In the Toeplitz graph $T_9\langle 2,3\rangle$ the Hamiltonian path connecting 1 and 9 is shown in Fig. 10. The Hamiltonian path connecting 1 and 5k + 4 for any $k \geq 2$ in the Toeplitz graphs $T_{5k+4}\langle 2,3\rangle$ is obtained by joining the path of Fig. 10 with the path of Fig. 7.

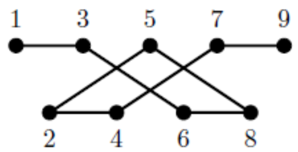

Figure 10: Hamiltonian path connecting 1 and 9 in $T_9\langle 2,3\rangle$

Case 5. If n = 0 (mod 5). The Hamiltonian path connecting 1 and 10 in Toeplitz graph $T_{10}\langle 2,3\rangle$ is shown in Fig.

11 while the Hamiltonian path connecting 1 and 5k for $k \geq 3$ in the Toeplitz graphs $T_{5k}\langle 2,3 \rangle$ is obtained by joining paths of Fig. 11 and Fig. 7.

Corollary 2.6. In Toeplitz graphs $T_n\langle 2,3 \rangle$, the Hamiltonian path $H[l,m]$ exists for $m-l \geq 5$ and $m-l \neq 7$.

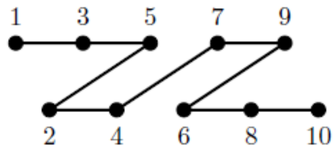

Figure 11: Hamiltonian path connecting 1 and 10 in $T_{10}\langle 2,3 \rangle$

Corollary 2.7. The Toeplitz graph $T_n\langle 2,3 \rangle$ for n ≥ 9 and $n \neq 11$ has a path with ends 1 and n that contains all vertices of the graph except vertices 2 and 3.

Proof. The required path is "$1, 4, H[4, n], n$".

Lemma 2.8. The Toeplitz graphs $T_n\langle 2,3 \rangle$ for all $n \geq 7$ has a paths $P'[1,n,2]$ and $P'[1,n,n-1]$.

Proof. The required paths for $T_7\langle 2,3 \rangle$ and $T_{10}\langle 2,3 \rangle$ are shown in Fig. 12. The required path for all other n is obtained by first connecting 1 with 3 and then connecting 3 with n using the Hamiltonian path of Lemma 2.5. Due to symmetry of Toeplitz graph there is a path with ends 1 and n that contains all vertices except $n-1$.

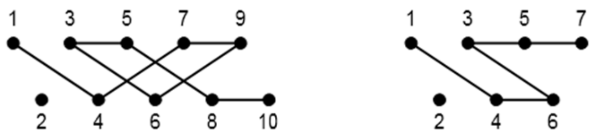

Figure 12: Path connecting 1 and n except 2

Corollary 2.9. In $T_n\langle 2,3 \rangle$ the path $P'[l,m,k]$ exists for $m-l \geq 6$.

Lemma 2.10. In $T_n\langle 2,3 \rangle$ for n ≥ 6 and $n \neq 7$ 7 the Hamiltonian paths $H_2^n[1,n]$ and $H_1^{n-1}[1,n]$ exists.

Proof. In $T_6\langle 2,3 \rangle$ the required Hamiltonian path is "$2,5,3,1,4,6$". The path "$2,5,7,4,1,3,6,8$" is the required Hamiltonian path in $T_8\langle 2,3 \rangle$. The required Hamiltonian path in $T_9\langle 2,3 \rangle$ is "$2,4,1,3,6,8,5,7,9$". In $T_n\langle 2,3 \rangle$ for $n \geq 9$ the required path is "$2,4,1,3,P'[3,n,4],n$".

Lemma 2.11. In $T_n\langle 2,3,t \rangle$, $n \geq 20$ and $4 \leq t \leq n-11$, the Hamiltonian path $H_1^x[1,n]$ exists for all $2 \leq x \leq n$.

Proof. The Hamiltonian paths between vertex 1 with all other vertices except $n-3$ $4 \le t \le n-7$ are shown in Table 1.

Table 1: Hamiltonian Path between 1 and other vertices.

End Vertices	Hamiltonian Path
$1-2$, $(n \ge 6)$	$1, P[1,2,n], 2.$
$1-3$, $(n \ge 10)$	$1,4,2,5,P[5,6,n],6,3.$
$1-4$, $(n \ge 10)$	$1,3,6,P[5,6,n],5,2,4.$
$1-5$, $(n \ge 11)$	$1,3,6,P[6,7,n],7,4,2,5.$
$1-x$, $(n \ge 11)$ $6 \le x \le n-5$	$1,\ P'[1,x+1,x],x+1,\ P[x,x+1,n],x.$
$1-(n-4)$, $(n \ge 14)$	$1,H[1,n-5],n-5,P[n-5,n-4,n],n-4.$
$1-(n-2)$, $(n \ge 13)$	$1,H[1,n-4],n-4,n-1,n-3,n,n-2.$
$1-(n-1)$, $(n \ge 13)$	$1,H[1,n-4],n-4,n-2,n,n-3,n-1.$
$1-n$, $(n \ge 9)$	$1,H[1,n],n.$

Hamiltonian Path Between 1 and $(n-3)$ in $T_n\langle 2,3,t\rangle$, for different values of t is shown in Table 2.

Table 2: Hamiltonian Path between 1 and $n-3$.

Condition on *t*	Hamiltonian Path
$t=4, (n \ge 14)$	$1,H[1,n-5],n-5,n-1,n-4,n-2,n,n-3.$
$t=5$, $(n \ge 14)$	$1,H[1,n-5],n-5,n,n-2,n-4,n-1,n-3.$
$t=6$, $(n \ge 13)$	$1,\ P'[1,n-6,n-7],n-6,n,n-2,n-5,n-7,n-4,n-1,n-3.$
$t=7$, $(n \ge 14)$	$1,P^{l}[1,n-7,n-8],n-7,n,n-2,n-5,n-8,n-6,n-4,n-1,n-3.$
$t=8$, $(n \ge 15)$	$1,P^{l}[1,n-8,n-9],n-8,n,n-2,n-5,n-7,n-9,n-6,n-4,n-1,n-3.$
$9 \le t \le n-11$, $(n \ge 20)$	$1,H[1,n-t-2],n-t-2,P[n-t-2,n-t-1,n-5],n-t-1,n-1,n-4,$ $n-2,n,n-3.$ This path is shown in Fig. 13.

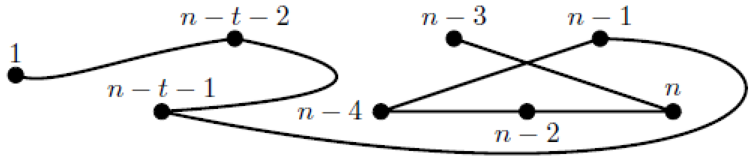

Figure 13: Hamiltonian path with ends 1 and $n-3$ in $T_n\langle 2,3,\mathrm{t}\rangle$ when $4 \le t \le n-6$

Lemma 2.12. In $T_n\langle 2,3,t\rangle$, $n \ge 18$ and $4 \le t \le n-10$, the Hamiltonian path $H_1^x[1,n]$ exists for all $1 \le x \le n$

$x \neq 2$.

Proof. The Hamiltonian path of 2 with 1 is proved in Lemma 2.11 and with all other vertices except $n-3$ is shown in Table 3.

Table 3: Hamiltonian Path of 2 with other vertices.

End Vertices	Hamiltonian Path
2 – 3, $(n \geq 9)$	$3, 1, 4, P[4, 5, n], 5, 2.$
2 – 4, $(n \geq 10)$	$2, 5, P[5, 6, n], 6, 3, 1, 4.$
2 – 5, $(n \geq 10)$	$2, 4, 1, 3, 6, P[5, 6, n], 5.$
2 – 6, $(n \geq 10)$	$2, 4, 1, 3, 5, P[5, 6, n], 6.$
2 – 7, $(n \geq 11)$	$2, 5, 3, 1, 4, 6, P[6, 7, n], 7.$
2 – x, $(n \geq 14)$ $9 \leq x \leq n-4$	$2, H_2^{x-1}[1, x-1], x-1, P[x-1, x, n], x.$
2 – $(n-1)$, $(n \geq 13)$	$2, H_2^{n-5}[1, n-5], n-5, n-3, n, n-2, n-4, n-1.$
2 – $(n-2)$, $(n \geq 12)$	$2, H_2^{n-4}[1, n-4], n-4, n-1, n-3, n, n-2.$
2 – n, $(n \geq 8)$	$2, H_2^{n}[1, n], n.$

The Hamiltonian path of 2 with $n-3$ for $4\,4 \leq t \leq n-10$ is shown in Table 4.

Table 4: Hamiltonian Path of 2 with $n-3$

Condition on t	Hamiltonian Path
$t = 4$, $(n \geq 13)$	$2, H_2^{n-5}[1, n-5], n-5, n-1, n-4, n-2, n, n-3.$
t = 5, $(n \geq 13)$	$2, H_2^{n-5}[1, n-5], n-5, n, n-2, n-4, n-1, n-3.$
t = 6, $(n \geq 16)$	$2, 4, 1, 3, P[3, n-6], n-6, n, n-2, n-5, n-7, n-4, n-1, n-3$
$t = 7$, $(n \geq 17)$	$2, 4, 1, 3, P[3, n-7], n-7, n, n-2, n-5, n-8, n-6, n-4, n-1, n-3.$
$8 \leq t \leq n-10$, $(n \geq 18)$	$2, H_2^{n-t-2}[1, n-t-2], n-t-2, P[n-t-2, n-t-2\ 1, n-5], n-t-1, n-1, n-4, n-2, n, n-3.$

Lemma 2.13. In $T_n\langle 2,3,t\rangle$, $n \geq 20$ and $4 \leq t \leq n-11$, the Hamiltonian path $H_3^x[1,n]$ exists for all $1 \leq x \leq n$ and $x \neq 3$.

Proof. The concerned Hamiltonian path of 3 with 1 and 2 are shown in Lamma's 2.11 and 2.12 respectively. Hamiltonian path of 3 with 4 is shown in Table 5.

Table 5: Hamiltonian Path of 3 with 4

Condition on t	Hamiltonian Path
$t=4, \quad (n\geq 10)$	$3,1,5,P[5,6,n],6,2,4.$
$t=5, \ (n\geq 11)$	$3.5,2,7,P[6,7,n],6,1,4.$
$t=6, \quad (n\geq 14)$	$3,5,2,8,6,9,P[9,10,n],10,7,1,4.$
$t=7, \ (n\geq 14)$	$3,1,8,6,9,P[9,10,n],10,7,1,4.$
$t=8,\ n\geq 20$	$3, 1, 9, 7, 15, P[15, 16, n], 16, 13, 11, 14, 12, 10, 2, 5, 8, 6, 4.$
$9 \leq t \leq n-7$ $t = 1 \pmod 2$ $(n\geq 16)$	$3,1,t+1,t-2,t,t+3,P[t+2,t+3,n],t+2,2,5,7,\ldots,t-4,t-1,t-3,\ldots,8,6,4.$
$10 \leq t \leq n-7$ $t = 0 \pmod 2$ $(n\geq 17)$	$3,1,t+1,t-2,t-5,t-7,\ldots,5,2,t+2,P[t+2,t+3,n],t+3,t,t-3,t-1,$ $t-4,t-6,\ldots,8,6,4.$

The Hamiltonian path between 3 and 5 for $4\leq t\leq n-8$ is shown in Table 6.

Table 6: Hamiltonian Path of 3 with 5

Condition on t	Hamiltonian Path
$t=4,\ (n\geq 11)$	$3,1,4,7,P[6,7,n],6,2,5.$
$t = 0 \pmod 2$ $6 \leq t \leq n-7\ (n\geq 13)$	$3,1,t+1,t-1,t-3,\ldots,7,4,6,8,\ldots,t,t+3,P[t+2,t+3,n],t+2,2,5.$
$t = 1 \pmod 2$ $5 \leq t \leq n-8\ (n\geq 13)$	$3,1,t+1,t+4,P[t+3,t+4,n],t+3,t,t-2,\ldots,7,4,6,8,\ldots$ $,t-1,t+2,2,5.$

The Hamiltonian paths of 3 with vertices in the set $\{6,7,8,\ldots,n\}$ except $n-3$ are shown in Table 7.

Table 7: Hamiltonian Path of 3 with 5.

End Vertices	Hamiltonian Path
$3-6,\ (n\geq 10)$	$3,1,4,2,5,P[5,6,n],6.$
$3-7,\ (n\geq 14)$	$3,1,4,2,5,H_5^7[5,n],7.$
$3-8,\ (n\geq 14)$	$3,1,4,2,5,\ H_5^8[5,n],8.$
$3-9,\ (n\geq 15)$	$3,1,4,2,5,\ H_5^9[5,n],9.$
$3-10,\ (n\geq 16)$	$3,1,4,2,5,\ H_5^{10}[5,n],10.$
$3-x,\ (n\geq 16)$ $11\leq x\leq n-5$	$3,1,4,2,5,P'[5,x+1,x],x+1,P[x,x+1,n],x.$
$3-n,\ (n\geq 17)$	$3,1,4,2,5,H[5,n],n.$
$3-(n-1),\ (n\geq 13)$	$3,1,4,2,5,P'[5,n-2,n-3],n-2,n,n-3,n-1.$
$3-(n-2),\ (n\geq 17)$	$3,1,4,2,5,H[5,n-4],n-4,n-1,n-3,n,n-2.$
$3-(n-4),\ n\geq 18$	$3,1,4,2,5,H[5,n-5],n-5,n-2,n,n-3,n-1,n-4.$

The Hamiltonian path between 3 and $n-3$ for $4 \le t \le n-7$ is given in the Table 8.

Table 8: Hamiltonian Path between 3 and $n-3$

Condition on t	**Hamiltonian Path**
$t=4$, $(n \ge 18)$	$3,1,4,2,5,H[5,n-5],n-5,n-1,n-4,n-2,n,n-3$
$t=5$, $(n \ge 18)$	$3,1,4,2,5,H[5,n-5],n-5,n,n-2,n-4,n-1,n-3.$
$t=6$, $(n \ge 17)$	$3,1,4,2,5,P'[5,n-6,n-7],n-6,n,n-2,n-5,n-7,n-4,n-1,n-3.$
$t=7$, $(n \ge 18)$	$3,1,4,2,5,P'[5,n-7,n-8],n-7,n,n-2,n-5,n-8,n-6,n-4,n-1,n-3.$
$t=8$, $(n \ge 19)$	$3,1,4,2,5,P'[5,n-8,n-9],n-8,n,n-2,n-5,n-7,n-9,n-6,n-4,n-1,n-3.$
$9 \le t \le n-11$, $(n \ge 20)$	$3,1,4,2,5,P'[5,n-t,n-t-1],n-t,P[n-t,n-t-1,n-5],n-t-1,n-1,n-4,$ $n-2,n,n-3.$

□

Lemma 2.14. In $T_n\langle 2,3,t\rangle$, $n \ge 24$ and $4 \le t \le n-11$, the Hamiltonian path $H_5^x[1,n]$ exists for all $1 \le x \le n$ and $x \ne 5$.

Proof. The Hamiltonian path of 5 with vertices 1, 2 and 3 is already shown. The Hamiltonian path of vertex 4 with 5 for different values of t is given in Table 9.

Table 9: Hamiltonian Path of Vertex 4 with vertex 5

Values of t	**Hamiltonian Path**
$t=4$, $(n \ge 11)$	$4,1,3,7,P[6,7,n],6,2,5.$
$t=5$, $(n \ge 11)$	$4,2,7,P[6,7,n],6,1,3,5.$
$t=6$, $(n \ge 14)$	$4,2,8,6,9,P[9,10,n],10,7,1,3,5.$
$t=7$, $(n \ge 15)$	$4,7,10,P[10,11,n],11,8,1,3,6,9,2,5.$
$t=8$, $(n \ge 15)$	$4,2,10,P[10,11,n],11,8,6,3,1,9,7,5.$
$t=9$, $(n \ge 17)$	$4,1,3,12,P[12,13,n],13,10,P[10,11,6],11,2,5.$
$10 \le t \le n-7$ $(n \ge 17)$	$4,2,t+2,P[t+2,t+3,n],t+3,t,P[t,t+1,6],t+1,1,3,5.$

The Hamiltonian path of vertex 5 with vertex 6 in Toeplitz graph $T_n\langle 2,3,\mathrm{t}\rangle$ for all values of $4 \le t \le n-7$ and $n \ge 14$ is given in Table 10.

Table 10: Hamiltonian Path of Vertex 5 with vertex 6

Conditions on t	**Hamiltonian Path**
$t=4$, $(n \ge 11)$	$5,2,4,1,3,7,P[6,7,n],6.$
$t=5$, $(n \ge 11)$	$5,3,1,4,2,7,P[6,7,n],6.$
$t=6$, $(n \ge 12)$	$5,3,1,7,P[7,8,n],8,2,4,6.$
$t=7$, $(n \ge 15)$	$5,3,1,8,11,P[10,11,n],10,7,9,2,4,6.$
$t=8$, $(n \ge 16)$	$5,8,11,P[11,12,n],12,9,7,10,2,4,1,3,6.$

$9 \le t \le n-7$, $(n \ge 16)$	$5,3,1,4,2,t+2,P[t+2,t+3,n],t+3,P^{l}[6,t+3,t+2],6.$

The Hamiltonian path of vertex 5 with vertex x such that $7 \le x \le n$ and $x \ne n-3$ in Toeplitz graph $T_n\langle 2,3,t\rangle$ for $4 \le t \le n-7$ and $n \ge 25$ is "$5,2,4,1,3,6, H_6^x[6,n],, x$ "

Main Result

Theorem 3.1. The Toeplitz graph $T_n\langle 2,3,\mathrm{t}\rangle$ is Hamiltonian connected for all $\mathrm{n} \ge 28$ and $4 \le t \le \frac{n-8}{2}$

Proof. For every $x, y \in V(T_n\langle 2,3,\mathrm{t}\rangle)$, and $6 \le x < y \le n-5$ the Hamiltonian path for different conditions on $y-x,$ for $4 \le t \le n-7$ and for $n \ge 17$ is shown in Table 11.

Table 11: Hamiltonian Path between x and y when $6 \le x < y \le n-5$

Conditions on x and y	Hamiltonian Path
$y-x \ge 5$ and $n \ge 16$	$x,\ P[x-1,x,1],\ x-1,P[x-1,y+1],\ y+1,\ P[y,y+1,n],\ y.$
$y-x=1$ and $n \ge 12$	$x,P[x-1,x,1],x-1,y+1,P[y,y+1,n],y.$
$y-x=2$ and $n \ge 13$	$x,P[x-1,x,1],x-1,x+1,y+1,P[y,y+1,n],y.$
$y-x=3$ and $n \ge 14$	$x,P[x,x+1,1],x+1,y+1,x+2,y+2,P[y+2,y+3,n],y+3,y.$
$y-x=4$ and $n \ge 16$	$x,P[x-1,x,1],x-1,x+2,y+1,P[y+1,y+2,n],y+2,x+3,x+1,y.$

The Hamiltonian path of vertex 4 with vertex 6 in Toeplitz graph $T_n\langle 2,3,t\rangle$ for all values of $4 \le t \le n-7$ is given in Table 12.

Table 12: Hamiltonian Path of Vertex 4 with vertex 6

Values of t	Hamiltonian Path
$t=4,\ (n \ge 12)$	$4,1,3,7,P[7,8,n],8,5,2,6.$
$t=5,\ (n \ge 11)$	$4,1,3,5,2,7,P[6,7,n],6.$
$t=6,\ (n \ge 11)$	$4,2,5,3,1,7,P[6,7,n],6.$
$t=7,\ (n \ge 12)$	$4,2,5,7,P[7,8,n],8,1,3,6.$
$t=8,\ (n \ge 16)$	$4,2,5,7,H_7^9[7,n],9,1,3,6.$
$9 \le t \le n-7,\ (n \ge 16)$	$4,1,3,5,2,t+2,P[t+2,t+3,n],t+3,\ P'[6,t+3,t+2],6.$

The Hamiltonian path of vertex 4 with vertex 7 in Toeplitz graph $T_n\langle 2,3,t\rangle$ for all values of $4 \le t \le n-7$ is given in Table 13.

The Hamiltonian path of vertex 4 with vertex 8 in Toeplitz graph $T_n\langle 2,3,t\rangle$ for all values of $4 \le t \le n-7$ is given in Table 14.

The Hamiltonian path of vertex 4 with vertex 9 in Toeplitz graph $T_n\langle 2,3,t\rangle$ for all values of $4 \le t \le n-7$ is given in Table 15.

Table 13: Hamiltonian Path of Vertex 4 with vertex 7

Values of t	**Hamiltonian Path**
$t=4$, $(n \ge 11)$	$4,1,3,5,2,6,P[6,7,n],7.$
$t=5$, $(n \ge 11)$	$4,2,5,3,1,6,P[6,7,n],7.$
$t=6$, $(n \ge 14)$	$4,1,3,5,2,8,6,9,P[9,10,n],10,7.$
$t=7$, $(n \ge 14)$	$4,2,5,3,1,8,6,9,P[9,10,n],10,7.$
$t=0 \pmod 2$ $8 \le t \le n-7$ $(n \ge 5)$	$4,2,t+2,P[t+2,t+3,n],t+3,t,t-2,\ldots,8,6,9,11,\ldots,t+1,1,3,5,7.$
$t=1 \pmod 2$ $9 \le t \le n-7$ $(n \ge$	$4,2,t+2,P[t+2,t+3,n],t+3,t,t-2,t-4,\ldots,9,6,8,10,\ldots,t+1,1,3,5,7.$

Table 14: Hamiltonian Path of Vertex 4 with vertex 8

Values of t	Hamiltonian Path
$t=4$, $(n \ge 15)$	$4,1,3,5,2,6,9,7,10,P[10,11,n],11,8.$
$t=5$, $(n \ge 15)$	$4,2,5,3,1,6,9,7,10,P[10,11,n],11,8.$
$t=6$, $(n \ge 15)$	$4,2,5,3,1,7,10,P[9,10,n],9,6,8.$
$t=7$, $(n \ge 18)$	$4,1,3,5,2,9,H_9^8[6,n],8.$
$t=8$, $(n \ge 14)$	$4,7,5,2,10,P[9,10,n],9,1,3,6,8.$
$t=1 \pmod 2$ $9 \le t \le n-7$, $(n \ge 16)$	$4,1,t+1,t-1,t-3,\ldots,10,7,9,11,\ldots,t,t+3,P[t+2,t+3,n],t+2,2,5,3,6,8$
$t=0 \pmod 2$ $10 \le t \le n-7$, $(n \ge 17)$	$4,1,t+1,t-1,t-3,\ldots,7,10,12,\ldots,t,t+3,P[t+2,t+3,n],t+2,2,5,3,6,8.$

Table 15: Hamiltonian Path of Vertex 4 with vertex 9

Values of t	**Hamiltonian Path**
$t=4$, $(n \ge 15)$	$4,1,3,5,2,6,H_6^9[6,n],9.$
$t=5$, $(n \ge 15)$	$4,1,3,5,2,7,H_7^9[6,n],9.$
$t=6$ $(n \ge 19)$	$4,1,3,5,2,8,H_8^9[6,n]],9.$
$t=7$, $(n \ge 19)$	$4,2,5,3,1,8,H_8^9[6,n],9.$
$t=8$, $(n \ge 15)$	$4,1,3,6,8,11,P[10,11,n],10,2,5,7,9.$
$t=9$, $(n \ge 15)$	$4,1,3,6,8,10,P[10,11,n],11,2,5,7,9.$
$t=1 \pmod 2$ $11 \le t \le n-7$ $(n \ge$	$4,7,5,2,t+2,P[t+2,t+3,n],t+3,t,t-2,t-4,\ldots,11,8,10,12,\ldots,t+1,1,3,6,9.$
$t=0 \pmod 2$ $10 \le t \le n-7$ $(n \ge$	$4,7,5,2,t+2,P[t+2,t+3,n],t+3,t,t-2,t-4,\ldots,8,11,13,15,\ldots,t+1,1,3,6,9.$

The Hamiltonian path of vertex 4 with vertex 10 in Toeplitz graph $T_n\langle 2,3,t\rangle$ for all values of $4 \le t \le n-7$ is given in Table 16.

Table 16: Hamiltonian Path of Vertex 4 with vertex 10

Condition on *t*	**Hamiltonian Path**
$4 \le t \le n-7, t \ne 8, (n \ge 22)$	$4, 1, 3, 5, 2, t+2, H_{10}^{t+2}[6,n], 10.$
$t = 8, (n \ge 20)$	$4,2,5,3,1,9, H_9^{10}[6,n], 10.$

The Hamiltonian path of vertex 4 with vertex x for $11 \le x \le n-5$ in Toeplitz graph $T_n\langle 2,3,t\rangle$ for all values of $4 \le t \le \frac{n-8}{2}$ and $n \ge 27$ is given in Table 17.

Table 17: Hamiltonian Path of Vertex 4 with vertex $11 \le x \le n-5$

Condition on *t*	**Hamiltonian Path**
$t = 4, (n \ge 24)$	$4,1,3,5,2,6, H_6^x[6,n], x.$ For all $7 \le x \le n.$
$t = 5, (n \ge 24)$	$4,2,5,3,1,6, H_6^x[6,n], x.$ For all $7 \le x \le n.$
$t = 6, (n \ge 24)$	$4,2,5,3,1,7, H_7^x[6,n], x.$ For all $8 \le x \le n.$
$t = 7, (n \ge 25)$	$4,2,5,3,1,8, H_8^x[6,n], x.$ For all $9 \le x \le n.$
$t = 8, (n \ge 26)$	$4,1,3,5,2,10, H_{10}^x[6,n], x.$ For all $11 \le x \le n.$
$t = 9, \ (n \ge 26)$	$4,2,5,3,1,10, H_{10}^x[6,n], x.$ For all $11 \le x \le n.$
$10 \le t \le n-7, x = t+1, (n \ge 21)$	$4,1,3,5,2,t+2, h_{t+2}^x[6,n], x.$ For all $11 \le x \le n-5.$
$10 \le t \le n-7, x \ne t+1, (n \ge 21)$	$4,2,5,3,1,t+1, h_x^{t+1}[6,n], x$ Hamiltonian path of Table 11,

The Hamiltonian path of vertex 4 with vertex $n-3$, for $10 \le t \le \frac{n-8}{2}$ and for $n \ge 28$ is "$4,1,3,5,2, t+2,$ $P[t+2,t+1,6], t+1, P'[t+1, n-t-1, t+2], n-t-1, P[n-t-1, n-t, n-5], n-t, n, n-2, n-4, n-1, n-3$".

2. Conclusion

S. Malik and T. Zamfirescu investigated Hamiltonian properties of Toeplitz graphs in [8]. M. F. Nadeem Ayesha Shabbir and Tudor Zamfirescu complete the picture of Toeplitz graphs $T_n\langle t_1,t_2\rangle$, $T_n\langle 1,3,t\rangle$ and $T_n\langle 1,5,t\rangle$ in [9]. In this paper we proved that the Toeplitz graphs $T_n\langle 2,3,t\rangle$ are Hamiltonian connected for $n \ge 28$ and $4 \le t \le \frac{n-8}{2}$. It would be interesting to derive similar results for other families of Toeplitz graphs such as $T_n\langle 2,5,t\rangle$ and generalize the results for $T_n\langle 2,s,t\rangle$.

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