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On Treating of Surface Cracks in Finite Layers of Fractional Materials

Hassan Khayyat*

Faculty of Engineering,Mechanical Engineering Department Shaqra University,Saudi Arabia

E-mail: hkhayyat@su.edu.sa

Abstract

The plane strain problems of the bounded layer medium composed of three different materials contain a crack on one of the interface are considered. Using Fourier integral transform, the boundary value problem leads to a mixed integral equation with Cauchy kernel in position and continuous kernel in time. In addition, using a quadratic numerical method we have a system of Fredholm integral equations with Cauchy kernel in position. Then, the Jacobi polynomials method, according to the index of integral equation**,** is used to solve the system of Fredholm integral equations. Moreover, the developing program is used to computing the approximate solution and the estimated error.

Keywords: Fracture mechanics; Fourier integral transform; System of Fredholm integral equations; Cauchy kernel; Jacobi polynomial.

1. Introduction

Gdoutos, in his work [1], stated that, fracture mechanics is based on the assumption that all engineering materials contain cracks from which failure starts. It is known that, cracks lead to high stresses near the crack growth takes place. In [2], Erdogen and Kaya studied the elasticity problem for an orthotropic strip or a beam with an internal or an edge crack under general loading condition. In [3], Erdogan discussed some different method of solution of elastic crack problem, and he described number of related special mechanics problems. In [4], Matbuly and Nassar analyzed the electrostatic problem of an edge cracked orthotropic strip. The crack possesses a semi-infinite length. Moreover, the crack surfaces are subjected to opening mode I fracture, by a concentrated force action.

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* Corresponding author.

More information for treating the open cracks of the fracture mechanics are found in [5-7]. In other side, many authors have interested in solving the fracture mechanics with curvilinear hole. Some authors expressed the solution of fractional mechanics materials in terms of two complex potential functions in the form of Laurent's series, see [8,9,10]. Other used complex variables method (Cauchy method) to obtain the solution of fracture mechanics with curvilinear hole in the form of two Gaursat functions, see [11-16]. In most of the problems of fractional mechanics materials, we obtain a singular integral equation. For this, many different methods numerical and analytic are established for solving the contact problems. More information for using the different methods for solving integral equations in fractional mechanics materials, see [17-21].

In fractional mechanics materials problem, the unknown function (say $\phi(x,t)$) may be either a potential or a flux-type quantity.

In this paper, under certain conditions, the problem of fractional mechanics materials of three layers, after using Fourier transforms, leads to mixed integral equation in the space $L_2\left[-1,1\right]\times C[0,T]$, $0\le t\le T<1.$ i.e, we will have

$$\mu\phi(x,t)-\frac{\lambda}{\pi}\int_{-1}^{1}\frac{\phi(y,t)}{y-x}dy-\lambda\int_{-1}^{1}k(x,y)\phi\left(y,t\right)dy-\lambda\int_{0}^{t}\xi(t,\tau)\phi(x,\tau)d\tau=f(x,t), \qquad (1.1)$$

The formula (1.1) is called mixed integral equation of the second kind in position and time, μ is a constant defined the kind of integral equation. If $\mu=0$, we have the mixed integral equation of the first kind, if $\mu\ne 0$, we have integral equation of the second kind. λ is a constant and has physical meaning. The given function $f(x,t)\in L_2\left[-1,1\right]\times C[0,T]$, and is called the free term. While $\phi(x,t)$ is the unknown function. The interval [-1, 1] is the domain of integration with respect to position *x*, and $\xi(t,\tau)$ is called the kernel of Volterra integral with respect to the time $t\in[0,T];T<1$. Then using Chebyshev-Jacobi polynomials according to the index of the integral equation, the solution of the integral equation is discussed at the index points. Moreover, some numerical methods, according to the index and the kind of Chebyshev- Jacobi, are considered and the error estimate, in each case, is computed.

2. Formulation of the generalized problem

Consider a plane strain problem of the bounded layer medium composed of three different materials, see Fig. (1.1). Let the medium material contains a crack on one of the interface. Without any loss in generality, the half-length of the crack is assumed unity. Consider with the effect of the ratio of the layer thickness to the crack length on the stress intensity factors and the strain energy release rate.

For interesting the disturbed stress state, whiles is variable also with time, caused by the crack. We assume that the overall stress distribution $\sigma_{ij}^{(0)}(x,y,t)$, in the imperfection free medium, is known. The stress state $\sigma_{ij}^{(1)}(x,y,t)$, in the cracked medium, may be expressed as

$$\sigma_{ij}^{(1)}(x,y,t)=\sigma_{ij}^{(0)}(x,y,t)+\sigma_{ij}(x,y,t), \quad i,j=x,y,z \tag{2.1}$$

Where, σ_{ij} is the disturbed state, which may be obtained by using the tractions

$$P_1(x,t)=-\sigma_{yy}^{(0)}(x,0;t);\ P_2(x,t)=-\sigma_{xy}^{(0)}(x,0;t), \quad |x|<1,\ t\in[0,T];T<1. \tag{2.2}$$

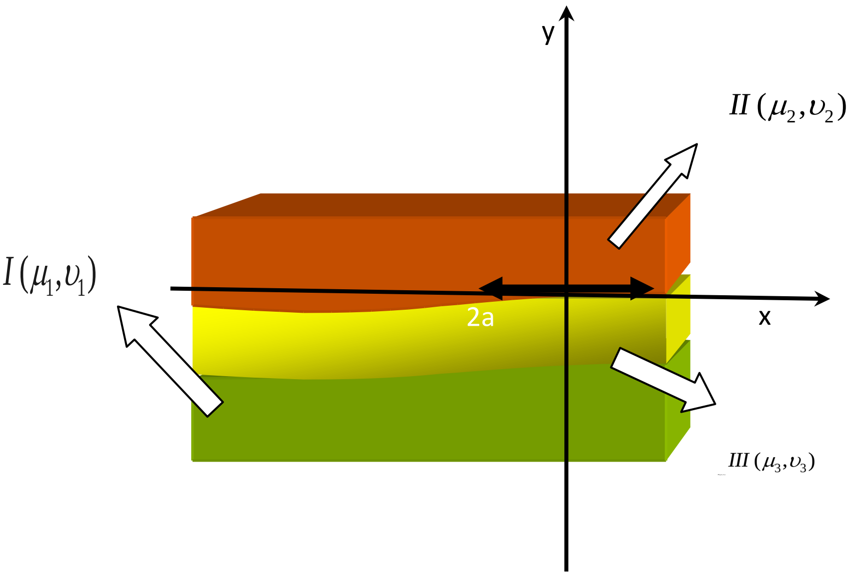

Figure 1.1

Which are the only external loads applied to the medium (the symmetry is considered with $(x=0)$. The general problem can always be expressed as the sum of a symmetric component and an anti-symmetric component. The tractions $P_i(x,t),\ (i=1,2)$, have the following properties

$$P_1(x,t)=P_1(-x,t), \quad P_2(x,t)=-P_2(-x,t),\ |x|<1, \quad t\in[0,T];T<1.. \tag{2.3}$$

The solution of the anti-symmetric problem requires only a slight modification. Let u_i, v_i be the x, y components of the displacement vector in the ith materials and satisfy the field equations in the form

$$\mu_i\nabla^2 u_i+(\lambda_i+\mu_i)\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\right)=\rho\frac{d^2u_i}{dt^2} \tag{2.4}$$

$$\mu_i\nabla^2 v_i+(\lambda_i+\mu_i)\frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\right)=\rho\frac{d^2v_i}{dt^2} \quad . \tag{2.5}$$

Then, assume the displacement functions in the form

$$u_i(x,y,t)=U_i(x,y)+F(t), \tag{2.6}$$

$$v_i(x,y,t)=V_i(x,y)+F(t), \tag{2.7}$$

where $F(t)$ is a function of *t, will be determined*.

Hence, using (2.6) and (2.7) in Eqs. (2.4) and (2.5), we have

$$(\lambda_i+2\mu_i)\frac{\partial^2 U_i}{\partial x^2}+\mu_i\frac{\partial^2 U_i}{\partial y^2}+(\lambda_i+\mu_i)\frac{\partial^2 V_i}{\partial x\,\partial y}=0, \tag{2.8}$$

$$(\lambda_i+2\mu_i)\frac{\partial^2 V_i}{\partial y^2}+\mu_i\frac{\partial^2 V_i}{\partial x^2}+(\lambda_i+\mu_i)\frac{\partial^2 U_i}{\partial x\,\partial y}=0, \tag{2.9}$$

and

$$\frac{d^2F(t)}{dt^2}=\frac{\mu_i}{\rho}F(t). \tag{2.10}$$

The formula (2.10) has a solution

$$F(t)=Be^{-\sqrt{\frac{\mu_i}{\rho}}t}, \qquad (F(\infty)\to 0). \tag{2.11}$$

In addition, for solving the two formulas (2.8) and (2.9), we use the Fourier integral transform, to obtain

$$U_i(x,y)=\frac{2}{\pi}\int_0^\infty\Big[(A_{i1}+A_{i2}y)e^{-\alpha y}+(A_{i3}+A_{i4}y)e^{\alpha y}\Big]\sin\alpha x\,d\alpha, \tag{2.12}$$

$$V_i(x,y)=\frac{2}{\pi}\int_0^\infty\left\{[A_{i1}+(\frac{K_i}{\alpha}+y)A_{i2}]e^{-\alpha y}+[-A_{i3}+(\frac{K_i}{\alpha}-y)A_{i4}]e^{\alpha y}\right\}\cos\alpha x\,d\alpha \tag{2.13}$$

Here, K_i have physical meaning, where $K_i=3-4\upsilon_i$ for plane strain and $K_i=(3-\upsilon_i)/(1+\upsilon_i)$ for generalized plane stress, υ_i are Poisson's coefficients for each materials, and $A_{i,j}$, $j=1,2,3,4$, are functions of α which can be determined from the boundary conditions. After obtaining the values of U_i, V_i, the stresses may be evaluated by Hook's law. In particular, the components of the stress vector at the interfaces and boundaries may be expressed, respectively as

$$\frac{1}{2\mu_i}\sigma_{yy}^i=\frac{2}{\pi}\int_0^\infty\{-\{\alpha(A_{i1}+A_{i2}y)+2(1-\upsilon_i)A_{i2}]e^{-\alpha y}+[-\alpha(A_{i3}+A_{i4}y)+2(1-\upsilon_i)A_{i4}]e^{\alpha y}\}\cos\alpha x\,d\alpha,$$

(2.14)

$$\frac{1}{2\mu_i}\sigma_{xy}^{i} = \frac{2}{\pi}\int_0^\infty \{-[\alpha(A_{i1}+A_{i2}y)+(1-2\upsilon_i)A_{i2}]e^{-\alpha y} + [\alpha(A_{i3}+A_{i4}y)-(1-2\upsilon_i)A_{i4}]e^{\alpha y}\}\sin \alpha x d\alpha.$$

(2.15)

On the boundaries, the medium may have formally any one of the following four groups of homogeneous boundary conditions

$$\text{(a)} \quad \sigma_{yy}^{i} = 0 = \sigma_{xy}^{i}\text{ , (b) } u_i = 0 = v_i\text{ , (c) } \sigma_{xy}^{i} = 0 = v_i\text{ , (d) }\sigma_{yy}^{i} = 0 = u_i\text{ , } \; i = 1,2,3 \quad (2.16)$$

The continuity requires that on the interfaces the stress and displacement vectors in the adjacent layers be equal i.e.

$$u_{i+1} - u_i = 0; v_{i+1} - v_i = 0; \sigma_{yy}^{i+1} - \sigma_{yy}^{i} = 0; \; \sigma_{xy}^{i+1} - \sigma_{xy}^{i} = 0 \text{ ,} \qquad (2.17)$$

Now, to represent the problem in the mixed integral equation form, we first assume that at $y = 0$ the bond between the two adjacent layers is perfect except for the (symmetrically located) dislocations at $y = 0, x = y$ defined by

$$\frac{\partial}{\partial x}\left(u_2^{+} - u_3^{-}\right) = f_1(x,t), \quad \frac{\partial}{\partial x}\left(v_2^{+} - v_3^{-}\right) = f_2(x,t)\text{ ,} \qquad (2.18)$$

In addition to (2.18), on the interface $y = 0,$ we have the following conditions

$$\sigma_{yy}^{2} - \sigma_{yy}^{3} = 0, \quad \sigma_{xy}^{2} - \sigma_{xy}^{3} = 0, \quad (0 \le x < \infty, \; y = 0)\text{ .} \qquad (2.19)$$

After some algebraic relations, the components of the stress vector at $y = 0$ and $x > 0$ may be expressed, respectively as

$$\frac{1+K_3}{\mu_3}\sigma_{yy}^{3}(x,0,t) = \lim_{y\to 0^-}\frac{2}{\pi}\int_0^\infty e^{\alpha x}\{a_{11}A_1(\alpha,t)+a_{12}A_2(\alpha,t)\}\cos \alpha x d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty \{H_{11}(\alpha)A_1(\alpha,t)+H_{12}(\alpha)A_2(\alpha,t)\}\cos \alpha x d\alpha + \int_0^t F(\tau) f_i(x,0,\tau)d\tau\text{ ,} \qquad (2.20)$$

$$-\frac{1+K_3}{\mu_3}\sigma_{xy}^{3}(x,0,t) = \lim_{y\to 0^-}\frac{2}{\pi}\int_0^\infty e^{\alpha x}\{a_{21}A_1(\alpha,t)+a_{22}A_2(\alpha,t)\}\sin \alpha x d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty \{H_{21}(\alpha)A_1(\alpha,t)+H_{22}(\alpha)A_2(\alpha,t)\}\sin\alpha x d\alpha+\int_0^t F(\tau)f_i(x,0,\tau)d\tau, \quad (2.21)$$

Where $H_{i,j}(\alpha)$ are the Heaviside functions and A_i are the Fourier transforms of f_i defined as follows

$$A_1(\alpha,t)=\int_0^\infty f_1(z,t)\cos\alpha z\, dz; \qquad A_2(\alpha,t)=\int_0^\infty f_2(z,t)\sin\alpha z dz \quad , \quad . \qquad (2.22)$$

The constants a_{ij} depend on the elastic properties of the materials adjacent to the crack only and are given by

$$a_{11}=-a_{22}=(1+\lambda_2\lambda_4)/\lambda_4, \qquad a_{12}=-a_{21}=-(1+2\lambda_4-\lambda_2\lambda_4)/\lambda_4,$$

$$\lambda_2=(K_2\mu_3-K_3\mu_2)/(\mu_2+K_2\mu_3), \qquad \lambda_4=(\mu_3+\mu_2K_3)/(\mu_2-\mu_3). \qquad (2.23)$$

Here, μ_i is the shear modulus and $\lambda's$ are Lame's constants.

The integrals on the right hand side are uniformity convergent. The formulas (2.20)-(2.22) give the stresses components for all values of x . The crack problem under consideration $f_i(x,t)$ are zero for $|x|>1$ and are unknown for $|x|<1$. On the other hand, the stress vector on the interface $y=0$ is unknown for $|x|>1$ and is given as known functions for $|x|<1$ i.e.

$$\sigma_{yy}^3(x,0,t)=P_1(x,t), \quad \sigma_{xy}^3(x,0,t)=P_2(x,t), \quad |x|<1. \qquad (2.24)$$

Using above information and the following symmetric properties in the presence of time $f_1(x,t)=f_1(-x,t); f_2(x,t)=-f_2(-x,t)$, we get

$$-\frac{1+K_3}{\mu_3}P_1(x,t)=\lim_{y\to 0^-}\left[\frac{a_{11}}{\pi}\int_{-1}^{1}f_1(z,t)dz\int_0^\infty e^{\alpha y}\cos\alpha(z-x)d\alpha+\frac{a_{12}}{\pi}\int_{-1}^{1}f_2(z,t)dz\,\times\right.$$

$$\left.\int_0^\infty e^{\alpha y}\sin\alpha(z-x)d\alpha+\frac{1}{\pi}\int_{-1}^{1}\sum_{j=1}^{2}k_{1j}(x,z)f_j(z,t)dz+\int_0^t F(\tau)f_1(x,\tau)d\tau\right],$$

and

$$-\frac{1+K_3}{\mu_3}P_3(x,t)=\lim_{y\to 0^-}\left[\frac{a_{21}}{\pi}\int_{-1}^{1}f_1(z,t)dz\int_0^\infty e^{\alpha y}\sin\alpha(z-x)d\alpha+\frac{a_{22}}{\pi}\int_{-1}^{1}f_2(z,t)dz\int_0^\infty e^{\alpha y}\cos\alpha(z-x)dx\right.$$

$$+\frac{1}{\pi}\int_{-1}^{1}\sum_{j=1}^{2}k_{2j}(x,z)f_j(z,t)dz+\int_{0}^{t}F(\tau)f_2(x,\tau)d\tau\Bigg]. \qquad (2.25)$$

Here, the bounded kernels k_{ij} are given by

$$k_{11}(x,z)=\int_{0}^{\infty}H_{11}(\alpha)\cos\alpha(z-x)d\alpha;\quad k_{12}(x,z)=\int_{0}^{\infty}H_{12}(\alpha)\sin\alpha(z-x)d\alpha\,,$$

$$k_{21}(x,z)=\int_{0}^{\infty}H_{21}(\alpha)\sin\alpha(z-x)d\alpha;\quad k_{22}(x,z)=\int_{0}^{\infty}H_{22}(\alpha)\cos\alpha(z-x)d\alpha\,, \qquad (2.26)$$

Evaluating the infinite integrals in (2.25), passing to the Cauchy theorem in complex analysis, we have

$$\frac{1+K_3}{a_{12}\mu_3}P_1(x,t)=\gamma\phi_1(x,t)+\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_2(y,t)dy}{x-y}-\frac{1}{a_{12}\pi}\int_{-1}^{1}\sum_{j=1}^{2}k_{1j}(x,y)\phi_j(y,t)dy+\frac{1}{a_{12}}\int_{0}^{t}F(\tau)\phi_1(x,\tau)dt \quad (2.27)$$

,

$$\frac{1+K_3}{a_{21}\mu_3}P_2(x,t)=\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_1(y,t)dy}{y-x}-\gamma\phi_2(x,t)-\frac{1}{a_{21}\pi}\int_{-1}^{1}\sum_{j=1}^{2}k_{2j}(x,y)\phi_j(y,t)dy+\frac{1}{a_{21}}\int_{0}^{t}F(\tau)\phi_2(x,\tau)d\tau \quad (2.28)$$

$$\gamma=\frac{a_{11}}{a_{12}}=\frac{a_{22}}{a_{21}}=\frac{(\mu_3+K_3\mu_2)-(\mu_2+K_2\mu_3)}{(\mu_2+K_2\mu_3)+(\mu_3+K_3\mu_2)}. \qquad (2.29)$$

The two formulas of (2.27), (2.28) represent a system of mixed integral equation with Cauchy kernel. For one layer, we write Eq. (2.27) in the following form

$$\eta\phi(x,t)-\frac{\lambda}{\pi}\int_{-1}^{1}p(|y-x|)\phi(y,t)dy-\lambda\int_{-1}^{1}k(x,y)\phi(y,t)dy-\lambda\int_{0}^{t}\xi(t,\tau)\phi(x,\tau)d\tau=f(x,t),$$
$$\{p(|y-x|)=\frac{1}{y-x}\}. \qquad (2.30)$$

The ends ± 1 are points of geometric singularity. At these points, and for all values of time, $\phi(x,t)$ is bounded, if it is a potential, and $\phi(x,t)$ has a singularity, if it is a flux-type quantity of system integral equations**.** This equation may be arising from the formulation of elasticity problems for the parallel layers compressed by stamps with arbitrary profile. If the contact between the parallel layers and the stamps is frictionless the corresponding constant η is zero and the related system integral equations is of the first kind, while, if the contact is perfect adhesion the related system integral equations is of the second kind, where the lengths of the cracks or size of the stamps are not equals.

3. The Existence of a unique solution of mixed integral equation:

In this section, we use Banach fixed-point theorem to prove the existence of a unique solution of mixed integral equation (2.30), under certain conditions. For this, we write (2.30) in the integral operator form

$$\bar{W}\phi(x,t)=\frac{1}{\eta}f(x,t)+W\phi(x,t), \quad (\eta\neq 0), \tag{3.1}$$

where

$$W\phi=H\phi+D\phi+\xi\phi, \tag{3.2}$$

and

$$H\phi=\frac{\lambda}{\eta\pi}\int_{-1}^{1}p(|y-x|)\phi(y,t)dy,\ D\phi=\frac{\lambda}{\eta}\int_{-1}^{1}k(x,y)\phi(y,t)dy;\quad \xi\phi=\frac{\lambda}{\eta}\int_{0}^{t}\xi(t,\tau)\phi(x,\tau)d\tau, \tag{3.3}$$

Then, we assume the following conditions

i) The two kernels of Fredholm integral term satisfies in $L_2[-1,1]$, respectively, for the constants L and M the following conditions:

(i-a) $|k(x,y)|\leq L$, (i-b) $\left[\int_{-1}^{1}\int_{-1}^{1}p^2(|y-x|)dydx\right]^{\frac{1}{2}}=M$,

ii) The kernel of Volterra integral term $\xi(t,\tau)$; in the space $C[0,T]$, is continuous and satisfies for a constant N, the condition $|\xi(t,\tau)|\leq N,\ \forall\ t,\ \tau\in[0,T]$.

iii) The given function $f(x,t)$ is continuous in the space $L_2[-1,1]\times C[0,T]$, and its norm is

$$\|f(x,t)\|=\max_{0\leq t\leq T}\int_{0}^{t}\left[\int_{-1}^{1}f^2(x,\tau)dx\right]^{\frac{1}{2}}d\tau=R,$$

Theorem 1.1: The mixed integral equation has a unique solution in the space $L_2[-1,1]\times C[0,T]$, under the condition

$$|\eta|>|\lambda|\left(\frac{M}{\pi}+\sqrt{2}T\,(L+N)\right);\ T=\max_{0\leq t\leq T}t\ , \tag{3.4}$$

Proof: To prove Theorem 1.1, we must prove the following lemmas.

Lemma 1.1: The integral operator $\overline{W}$ maps $L_2\left[-1,1\right]\times C[0,T]$ into itself.

Proof: From the formulas (3.1)-(3.3), the normality of the integral operator $H\phi, D\phi$ and $\xi\phi$ will take the forms

$$\|H\phi\| \le \left|\frac{\lambda}{\eta\pi}\right| M\,\|\phi\|;\ \|D\phi\| \le \left|\frac{\lambda}{\eta}\right|\sqrt{2}LT\,\|\phi\|;\ \|\xi\phi\| \le \left|\frac{\lambda}{\eta}\right|\sqrt{2}NT\,\|\phi\|,\ T = \max_{0\le t\le T} t\ , \qquad (3.5)$$

Hence, with the aid of condition (iv) and (3.5), we get

$$\|\overline{W}\phi\| \le \frac{R}{|\eta|} + \alpha\|\phi\|;\ \left(\alpha = \left|\eta^{-1}\lambda\right|\left(\frac{M}{\pi} + \sqrt{2}T\,(L+N)\right),\ T = \max_{0\le t\le T} t\right), \qquad . \qquad (3.6)$$

The inequality (3.6) yields that, the operator $\overline{W}$ maps the ball S_ρ in $L_2\left[-1,1\right]\times C[0,T]$ into itself, where

$$\rho = \frac{R}{|\ \eta\ |}\left(\frac{1}{1-\alpha}\right)\ . \qquad (3.7)$$

Since $0<\rho<1,\ R>0$, therefore we must have $(\alpha<1)$. Moreover, the inequality (3.7) involves that, the operators W and $\overline{W}$ are bounded

Lemma 1.2: The integral operator (3.1), under the condition (3.4) is continuous and contraction operator.

Proof: For the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in the Banach space $L_2\left(-1,1\right)\times C[0,T]$ the formula (3.1) after using the conditions (i),(ii) and (iii), then applying Cauchy – Schwarz inequality , yields

$$\|\overline{W}\phi_1 - \overline{W}\phi_2\| \le \alpha\|\phi_1-\phi_2\|,\ \left(\alpha = \left|\eta^{-1}\lambda\right|\left(\frac{M}{\pi} + \sqrt{2}T\,(L+N)\right)\right). \qquad (3.8)$$

Hence, $\overline{W}$ is a continuous operator in the space $L_2\left[-1,1\right]\times C[0,T]$, and under the condition $(\alpha<1)$, $\overline{W}$ is a contraction operator . From Lemma 1.1 and Lemma 1.2 and Banach fixed -point theorem, we can decide that the operator $\overline{W}$ has a unique fixed point which is the unique solution of Eq. (.2.30). Then Theorem 1.1 is completely proved.

4. The System of Fredholm Integral Equations

In order to discuss the solution of mixed integral equation (2.30), we use a quadratic numerical method to transform the mixed integral equation in position and time to system of Fredholm integral equations. For this, let

$t = t_i$, $i = 0,1,2,...,n$, and follow the work of Abdou and Mustafa [22]. Hence, the Volterra integral term, of (2.30) becomes

$$\int_0^{t_i} \xi(t_i,\tau)\phi(x,\tau)d\tau = \sum_{j=0}^{i} w_j \xi(t_i,t_j)\phi(x,t_j) + R(x,t_i). \qquad (4.1)$$

The values of *i* and the order of the truncation error R_i are depending on the number of derivatives of $\xi(t,\tau)$ for all $\tau \in [0,T]$, with respect to t and w_j is the weights, where $w_0 = \frac{1}{2}h_0$, $w_i = \frac{1}{2}h_i$ and $w_j = h_j$, $0 < j < i$, h denotes the constant step size for integration. Using (4.1) in (2.30) and then after using the following notations: $\phi_i(x) = \phi(x,t_i)$, $f_i(x) = f(x,t_i)$, $\xi_{i,j} = \xi(t_i,t_j)$, $i = 0,1,2,...,n$, $0 \le j \le i$, we get

$$\mu_i\phi_i(x) + \frac{\lambda}{\pi}\int_{-1}^{1} p(|y-x|)\phi_i(y)dy + \lambda\int_{-1}^{1} k(x,y)\phi_i(y)dy = \psi_i(x), \qquad (4.2)$$

where

$$\mu_i = (\eta + \lambda w_i \xi_{i,i}), \ \psi_i = f_i(x) - \lambda\sum_{j=0}^{i-1} w_j\left[\xi_{i,j}\phi_j(x)\right], \ i = 0,1,2,...,n. \qquad (4.3)$$

The formula (4.2) represents system of Fredholm integral equations of the second kind,

To prove the existence of a unique solution of (4.2) according to the Banach fixed- point theorem, we let *E* be the set of all continuous functions $\phi_p(x)$ in the space $L_2[-1,1]$, where $\Phi(x) = \{\phi_0(x), \phi_1(x), ..., \phi_p(x), ...\}$ and define the norm in the Banach space *E* by

$$\|\Phi\|_E = \max_i \|\phi_i(x)\|_{L_2(-1,1)} \text{ and } \|\Psi\|_E = \max_i \|\psi_i(x)\|_{L_2(-1,1)} .$$

Consider the following two conditions

(1) $\max_i \|f_i(x)\|_{L_2} \le Q$, (2) $\sum_{j=0}^{i-1} \max_j |w_j \xi_{i,j}| \le P$, (Q, P are constants).

Theorem 2 (without proof): If the conditions (i), (iii), of theorem 1.1 with the two conditions (1) and (2) are satisfied, then the formula (4.2) has a unique solution in the space *E*, under the condition: $|\lambda|\left(P + \sqrt{2}L + \frac{M}{\pi}\right) < |\eta^*|$, $(\eta^* = \max_i \mu_i \ \forall i)$.

5. Jacobi polynomials method for solving system of equations (4.2)

The solution of the problem (4.2) can be written as

$$\phi(y) = g(y)w(y), \quad w(x) = (1-x)^{\alpha}(1+x)^{\beta}. \tag{5.1}$$

where the unknown function $g(y)$ is regular on $-1 \le y \le 1$. w(x) is the fundamental function of Eq. (4.2). The singular behavior of the solution may be characterized by a fundamental function and the index of the problem. Where the index $K = -(\alpha+\beta)$; $K = -1, 0, 1$. More information for the importance of K can be founded in [23].

The relation between the values of K and the weight function and the unknown potential function can be discussed as the following:

(1): If $K = -1$. Then, $\alpha = \frac{1}{2} = \beta$. In this case, the unknown function $\phi(x)$ is bounded at both ends.

(2): If $K = 0$, this means physically that: $\alpha = -\frac{1}{2}; \beta = \frac{1}{2}$, or $\alpha = \frac{1}{2}; \beta = -\frac{1}{2}$ and the weight function, in each case, can be determined. In this case, the unknown function $\phi(x)$ is bounded at one end and has an integrable singularity at the other one. In addition, no extra condition is needed (the condition is the constant be zero), and the solution being a unique.

(3): If $K = 1$, then, $\alpha = -\frac{1}{2} = \beta$. The function $\phi(x)$ has singularities at both ends, and $\phi(x)$ must satisfy an additional condition

$$\int_{-1}^{1} \left[F(x)\right]\frac{dx}{w(x)} = 0, \qquad \mu\phi(x) + \frac{\lambda}{\pi}\int_{-1}^{1} k(|y-x|)\phi(y)dy = F(x) \tag{5.2}$$

Once, from the above discuss the unknown function, the fundamental function, $w(y)$ and he index K; $K = -1, 0, 1$ can be connected by the Jacobi polynomials $P_n^{(\alpha,\beta)}(y), (n = 0, 1, \cdots)$ in the form

$$\phi(y) = w(y)\sum_{n=0}^{\infty} c_n P_n^{(\alpha,\beta)}(y), \tag{5.3}$$

where $c_n, (n = 0, 1, ...)$ are undetermined constants.

Hence, we represent the solution of (2.30) in the form of (5.3). For this, we use the following famous relation, see Szegö [24]

$$\frac{1}{\pi}\int_{-1}^{1} w(y)P_n^{(\alpha,\beta)}(y)\frac{dy}{y-x} = \cot(\pi\alpha)w(x)P_n^{(\alpha,\beta)}(x) - \frac{2^{-(\alpha+\beta)}\Gamma(\alpha)\Gamma(n+\beta+1)}{\pi\Gamma(n+\alpha+\beta+1)} \times {}_2F_1\left(n+1,-n-\alpha-\beta;1-\alpha,\frac{1-x}{2}\right)$$

$$[\ (|x|<1),\ n=0,1,2,\cdots.\ ] \quad (5.4)$$

Here, ${}_2F_1\left(n+1,-n+K;1-\alpha;\frac{1-x}{2}\right)$ and $P_n^{\alpha,\beta}(x)$ are the Hyper geometric function and the Jacobi polynomials, respectively. Using the following relation (see Erdelyi [25])

$$P_{n-K}^{(-\alpha,-\beta)}(x) = \frac{\Gamma(n-K-\alpha+1)}{\Gamma(n-K+1)\Gamma(1-\alpha)}\, {}_2F_1\left(n+1,-n+K;1-\alpha;\frac{1-x}{2}\right). \quad (5.5)$$

The relation (5.4), yields

$$\frac{1}{\pi}\int_{-1}^{1} w(y)P_n^{(\alpha,\beta)}(y)\frac{dy}{y-x} = -\frac{\mu}{\lambda}w(x)P_n^{(\alpha,\beta)}(x) - \frac{2^{-K}\Gamma(\alpha)\Gamma(1-\alpha)}{\pi}P_{n-K}^{(-\alpha,-\beta)}(x) \quad (5.6)$$

Substituting from (5.6) into (4.2), after neglecting the suffix i and using the properties of the gamma function, we have

$$\sum_{n=0}^{\infty}\lambda c_n\left[-\frac{2^{-K}}{\sin(\pi\alpha)}P_{n-K}^{(-\alpha,-\beta)}(x) + \pi h_n(x)\right] = \psi(x), \quad (-1<x<1), \quad (5.7)$$

where,

$$h_n(x) = \int_{-1}^{1} w(y)P_n^{(\alpha,\beta)}(y)k(x,y)dy, \quad (-1<x<1). \quad (5.8)$$

The functional equation (5.7) represents an infinite linear algebraic system with unknown coefficients c_n . Expanding both sides of (5.7) in the form of Jacobi polynomials $P_k^{(-\alpha,-\beta)}(x);\ (k=0,1,...)$. Then, multiplying both sides by $\left(w(-\alpha,-\beta)P_k^{(-\alpha,-\beta)}(x)\right)$, ,and integrating the result from -1 to 1 and using the orthogonal relation, see Erdelyi [25]

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(x)w(x)dx = \begin{cases} 0 & n\neq k \\ \theta_k^{(\alpha,\beta)} & n=k \end{cases}, \quad (5.9)$$

where

$$\theta_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\ \frac{\Gamma(k+\alpha+1)\ \Gamma(k+\beta+1)}{k!\ \Gamma(k+\alpha+\beta+1)}, \quad (k=0,1,...). \tag{5.10}$$

Hence, after truncating the series (5.7), we have

$$-\frac{2^{-K}\lambda}{\sin\pi\alpha}\theta_k^{(-\alpha,-\beta)}c_{k+K} + \pi\sum_{n=0}^{N} d_{nk}c_n = F_k, \quad (k=0,1,...,N), \tag{5.11}$$

where

$$\left.\begin{aligned} d_{nk} &= \int_{-1}^{1} P_k^{(-\alpha,-\beta)}(x)w(-\alpha,-\beta,x)h_n(x)dx \\ F_k &= \int_{-1}^{1} P_k^{(-\alpha,-\beta)}(x)w(-\alpha,-\beta,x)\psi(x)dx \end{aligned}\right\} . \tag{5.12}$$

1) In the case $K = -1$, we note that the first term in the series (5.7) is a constant times $c_o P_1^{(-\alpha,-\beta)}(x)$. Hence, in solving (5.11) it can be formally assumed that $c_{-1}=0$. Also, from (5.7) to (5.12) it is seen that $P_0^{(-\alpha,-\beta)}(x)=1$ is the first equation obtained from (5.11) .

2) In the case K=0, there are no additional arbitrary constants or conditions, and (5.11) provides (N + 1) linear algebraic system for the unknown constants $c_0,...,c_N$.

3) In the case K = 1, the N + 1 equations given by (5.11) contains N + 2 unknown constants, $c_0,...,c_{N+1}$. The additional equation for a unique solution is provided by the equilibrium or compatibility condition by substituting from (5.10) and using the orthogonal condition, hence Eq. (1.7.21) yields:

$$c_0\theta_0(\alpha,\beta) = P\ . \tag{5.13}$$

where

$$\theta_0^{(\alpha,\beta)} = \int_{-1}^{1} w(t)dt = 2^{\alpha+\beta+1}\ \frac{\Gamma(\alpha+1)\ \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\ , \quad k=0. \tag{5.14}$$

6. Applications and Discussions

Here, we employ the procedures of the Jacobi polynomials methods to solve (2.30) with $\mu=1$ where the exact solution $\phi(x,t)=x^2t^2$, at times $T=0.5$ and 0.9. In addition, we consider the continuous kernel of Fredholm integral term $(\frac{1}{y-x})$, while the given function $\xi(t,\tau)=t\tau$, with $k=2$ (k is the division of the interval of time [0, *T*]. In this case, the free function $f(x,t)$ can be computed from Eq. (2.30). For the Polyurethane and the Fibber materials, we compute $\lambda = \dfrac{2G\upsilon}{(1-2\upsilon)}$, that corresponding to the value of Poison ratio $\upsilon = 0.22,\ 0.389$ and $G = 1\times10^7,\ 0.132\times10^7$, respectively. In our applications, we let *N* = 40, to get the approximate solutions, ϕ^P , and the corresponding error, E^P . In the first case, let K=0, and write in (2.30) the unknown function $\phi(x)$ in the Jacobi polynomials form

$$\phi(x) = \sum_{n=0}^{N} c_n w(x) P_n^{(\frac{1}{2},-\frac{1}{2})}(x), \quad w(x) = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}, \tag{6.1}$$

The unknown coefficients $c_1,\dots,c_N$ can be obtained from Eq. (5.11) and then substituted into Eq. (6.1) to get the approximate solution. Secondly, if we take *K*=1, then the unknown function is expressed in Cheypshev polynomials formula of the first kind as

$$\phi(x) = \sum_{0}^{N} c_n w(x) T_n(x), \qquad w(x) = (1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}} \ . \tag{6.2}$$

The system of **IEs** (5.11) may be solved, after determining the unknown coefficients $c_1,\dots,c_{N-1}$. In addition, the physics of the problem requires that the solution satisfy the compatibility conditions, which used to determine the *N* arbitrary constants in the solution. In the later case, we assume *K* = -1, then the unknown function $\phi(x)$ is expressed as

$$\phi(x) = \sum_{0}^{N} c_n w(x) U_n(x), \qquad w(x) = (1-x^2)^{\frac{1}{2}} \ . \tag{6.3}$$

In the following Tables (1-1) -(1-3), the results of the approximate solutions ϕ^P , ϕ^C and ϕ^U and the errors E^P, E^C and E^U , respectively, are obtained for different cases of time $T = 0.5$ and 0.9. In this part, we note the following results:

1. For $T = 0.5$ and 0.9, the values of the approximate solutions ϕ^P , ϕ^C and ϕ^U equal to zero at $x = \pm1$, or at $x = \pm1$, the values of the errors E^P, E^C and E^U are equal the exact solutions.

2. The values of E^C are less than the other errors E^P and E^U , where we have

$E^C < E^P < E^U$.

Case 1 : We apply the Jacobi polynomials method to solve Eq.(5.6)

Table 1-1

t	*x*	exact	Jacobi polynomials method			
			Fibber v=0.22		Polyurethane v=0.389	
			Appr.	error	Appr.	Error
0.5	-1.00E+00	2.500E-01	0.000E+00	2.500E-01	0.00E+00	2.500E-01
	-4.00E-01	4.000E-02	0. 409E-02	0.254E-04	2.60101E-02	0.399E-04
	-2.00E-01	1.000E-02	0.990E-02	0.858E-05	0.983E-02	0. 817E-05
	2.00E-01	1.000E-02	0.10047E-02	0. 053E-05	0.2490E-02	0.0 10E-05
	4.00E-01	4.000E-02	0. 428E-02	0.057E-05	0.853E-02	0. 011E-05
	1.00E+00	2.500E-01	0.000E+00	2.500E-01	0.000E+00	2.500E-01
0.9	-1.00E+00	0.990E-01	0.000E+00	0.990E-01	0.000E+00	0.990E-01
	-6.00E-01	2.916E-01	2.916E-01	1.300E-05	2.919E-01	1. 275E-05
	-2.00E-01	3.240E-02	3.240E-02	0.5 83E-05	3.240E-02	0.494E-05
	2.00E-01	3.240E-02	3. 241E-02	0.5 04E-05	3. 241E-03	0.5 04E-05
	6.00E-01	2.916E-01	2.917E-02	0.269E-04	2. 911E-02	0.310E-04
	1.00E+00	0.990E-01	0.000E+00	0.990E-01	0.000E+00	0.990E-01

Table (1-1) show the change of errors E^{p} of Fibber and Polyurethane materials at N = 40 for T=0.5,0.9

Case 2 : We apply Chebyshev polynomials of the first kind to solve Eq.(5.6).

Table 1-2

t	*x*	exact	Cheypshev polynomials method $T(x)$			
			Fibber v=0.22		Polyurethane v=0.389	
			Appr.	Error	Appr.	error
0.5	-1.000E+00	2.500E-01	0.000E+00	2.500E-01	0.000E+00	2.500E-01
	-6.000E-01	0. 600E-02	0.600E-02	0.690E-05	0.600E-02	0.490E-05
	-2.000E-01	1.000E-02	0.998E-02	0.884E-05	0.9999E-02	0.328E-05
	2.000E-01	1.000E-02	0.998E-02	0.690E-05	0.600E-02	0. 490E -05
	6.000E-01	0. 600E-02	0.600E-02	2.500E-01	9.058E-02	5.856E-04
	1.000E+00	2.50000E-01	0.000E+00	2.500E-01	0.000E+00	2.500E-01
0.9	-1.000E+00	8.100E-01	0.000E+00	8.100E-01	0.000E+00	8.100E-01
	-6.000E-01	2.916E-01	2.916E-01	0.351E-04	2.917E-01	0.1783E-04
	-2.000E-01	3.240E-02	3.244E-01	0.248E-04	3.243E-01	0.102E-04
	2.000E-01	3.240E-02	3.244E-01	0. 248E -04	3.243E-01	0.102E-04
	6.000E-01	2.916E-01	3.244E-01	0.248E-04	3.243E-01	0.102E-04
	1.000E+00	8.100E-01	0.000E+00	8.100E-01	0.000E+00	8.100E-01

Table (1-2) show the change of errors E^C of **Fibber** and Polyurethane materials at N = 40 for T= 0.5,0.9

Case (3) : We apply the Chebyshev polynomials of the second kind $U(x)$ to solve Eq.(5.6)

Table 1-3

t	***x***	**exact**	**Chebyshev polynomials method $U(x)$**			
			Fibber y v=0.22		**Polyurethane v=0.389**	
			Appr.	**error**	**Appr.**	**error**
0.5	**-1.000E+00**	**2.500E-01**	**0.000E+00**	**2.500E-01**	**0.000E+00**	**2.500E-01**
	-6.000E-01	**9.000E-02**	**0.600E-02**	**0.893E-04**	**0.600E-02**	**0.873E-04**
	-2.000E-01	**1.000E-02**	**0.995E-02**	**0.479E-05**	**0.996E-02**	**0.995E-05**
	2.000E-01	**1.000E-02**	**0.995E-02**	**0.479E-05**	**0.996E-02**	**0.873E-04**
	6.000E-01	**9.000E-02**	**0.600E-02**	**0.893E-04**	**0.600E-02**	**0.873E-04**
	1.000E+00	**2.500E-01**	**0.000E+00**	**2.500E-01**	**0.000E+00**	**2.500E-01**
0.9	**-1.000E+00**	**8.100E-01**	**0.000E+00**	**8.100E-01**	**0.000E+00**	**8.100E-01**
	-6.000E-01	**2.916E-01**	**2.916E-01**	**0.819E-04**	**2.916E-01**	**0.751E-04**
	-2.000E-01	**3.240E-02**	**2.240E-01**	**0.289E-04**	**2.240E-01**	**0.102E-04**
	2.000E-01	**3.240E-02**	**2.240E-01**	**0.289E-04**	**2.240E-01**	**0.102E-04**
	4.000E-01	**1.296E-01**	**1.296E-01**	**0.148E-04**	**1.334E-01**	**0.826E-04**
	6.000E-01	**2.916E-01**	**2.916E-01**	**0.819E-04**	**2.916E-01**	**0.751E-04**
	1.000E+00	**8.100E-01**	**0.000E+00**	**8.100E-01**	**0.000E+00**	**8.100E-01**

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