American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS)

ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

http://asrjetsjournal.org/

Extension of 2-Dimensional Planar Systems from Homological Algebra Perspective

Dr. Lewis Brew[a*], Joseph Acquah[b], Prof. Newton Amegbey[c]

[a,b]*University of Mines and Technology, Department of Mathematics, Tarkwa, Ghana*

[c]*University of Mines and Technology, Faculty of Engineering, Tarkwa, Ghana*

[a]*Email: amolewis@yahoo.com / lbrew@umat.edu.gh*

[b]*Email: jacquah@umat.edu.gh*

[c]*Email: na.amegbey@umat.edu.gh*

Abstract

This paper presents 2-Dimensional (2D) planar system (T_x) together with its changes in a topological space X as a dynamical system. Continuity which is one of the topological properties can be observed from the extension of systems. This paper therefore searches for something computable or an algebraic invariant to identify this topological property of 2D planar system (T_x). The approach is based on the notion of chains and cochains from homological algebra since the vertices and the edges are represented by the chain groups. The connectivity between different parallel chain complexes of the system is represented by the cochain groups. The extension of the 2-Dimensional planar system given by the homomorphism $F(b,x)$ for each parameter $b \in Z$ is the sequence of the cochain groups. The surface area of the system is represented by a cocycle and each cocycle is provided by a change in the parameter $b \in Z$. The dynamical properties of the system are studied by analysing different cocycle over it. The novel feature is the extension of the systems using the map $F(b,x)$, the cocycle and the 1st-Cohomology group which are detailed in the paper.

Keywords: Cochain groups; Chain groups; Cocycle; 1st-Cohomology group; Homotopy, Dynamical System; Exact sequence; Long Exact Sequence; 2-Dimensional (2D) planar system in the space X (T_x); Chain complex and Cochain complex.

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* Corresponding author.

1. Introduction

The study of dynamical systems involves a wide range of methods of analysing iterated mappings. These varied methods, which are due to developments in mathematics, have made it necessary to consider the study of dynamical system from other different perspective as carried out by this paper. In this paper, the issue of the extension or changes of 2D planar system is studied via the homomorphism $F : C^{b-1} \to C^b$ where C^* is the cochain map of the chain groups C_* of the system and F is the homomorphism of the cochain maps [2]. The concepts of cocycles, cochain complexes and the 1st cohomology group are used to study the connectivity of each extended system. Let X be a topological space and let $C_b(X)$ and $D_b(X)$ be two different horizontal parallel singular chain complexes in the space where each chain complex is connected together by a boundary ∂_b. If the horizontal chain complexes $C_b(X)$ and $D_{b+1}(X)$ are connected together by the cochain map $C^b : C_b(X) \to D_{b+1}(X)$, then the cochain complex consisting of group of homomorphisms $F : C^{b-1} \to C^b$ for $b \ge 0$ where $C^{b-1}\partial_b = \partial_{b+1}C^b$ provides a closed structure called a 2D planar system (T_x). The closed surface area exhibited by 2D planar system is homeomorphic to 2-cell and is called cocycle. The variable $b \ge 0$ outlines the changes in the 2D planar system at any time, as the homomorphism $F(b,x)$ continues to evolve. An abstract complex or system is a formal construction that builds a space in a combinatorial way through more simple objects. In this paper, the simple objects of a system are the vertices and the edges joining the vertices. The vertices correspond to 0-chain groups and the edges correspond to 1 chain groups [1]. Therefore 2D planar system inherits a cell complex with four vertices, four edges and one 2-cell and its extension over time $b \ge 0$ is determined through the homomorphism $F(b,x)$.

1.1 2D Planar Systems

Let the cochain $C^b : C_b(X) \to D_{b+1}(X)$ be a map of the chain complexes of the system in a topological space X such that the differentials ∂_b and ∂_{b+1} are the connecting homomorphism of the chain complexes C_b and D_{b+1}. If the two parallel chain complexes $\partial_b : C_b(X) \to C_{b-1}(X)$ and $\partial_{b+1} : \mathrm{D}_{b+1}(X) \to D_b(X)$ are connected together by cochain C^b such that $\mathrm{F} : C^{b-1} \to C^b$ is a map of cochain complexes and $C^{b-1}\partial_b = \partial_{b+1}C^b$ then the combination of chain and cochain complexes as illustrated at Figure 1.0 is a 2D planar system.

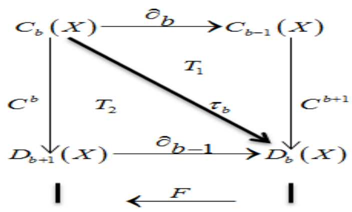

Figure 1: 2D Planar System

The long exact sequence of the 2D planar system increases the degree of the cochain groups. According to [3] the long exact sequence is another form of defining the extension of a short exact sequence. Thus, if $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence of the cochain complexes, then there exists natural maps $\partial : H^d(C) \to H^{d+1}(A)$ which provides the long exact sequence

$H^{d-1}(C) \xrightarrow{\partial} H^d(A) \xrightarrow{f} H^d(B) \xrightarrow{G} H^d(C) \xrightarrow{\partial} H^{d+1}(A) \longrightarrow$ as the extension of the short exact sequence.

Proposition 1.2 A 2D planar system is obtained if the chain complexes $\partial_b : C_b(X) \to C_{b-1}(X)$ and $\partial_{b+1} : \mathrm{D}_{b+1}(X) \to D_b(X)$ are connected together by cochain C^b such that there exists a map $\mathrm{F} : C^{b-1} \to C^b$ of cochain complexes and $C^{b-1}\partial_b = \partial_{b+1}C^b$.

Proof Let X be a space containing 2D planar system and $\partial_b : C_b(X) \to C_{b-1}(X)$ and $\partial_{b+1} : \mathrm{D}_{b+1}(X) \to D_b(X)$ be any two horizontal parallel chain complexes connected together by a cochain map C^b as shown at Figure 1.0. Let τ_b be a common boundary of the two triangular systems T_1 and T_2 .We see that each triangular system has a closed boundary. If the 2D planar system is oriented in a clockwise direction, then the boundary of T_1 thus $\partial(T_1)$ is

$$\partial(T_1) = \partial_b + C^{b-1} - \tau_b$$

Similarly

$$\partial(T_2) = \tau_b - \partial_{b+1} - C^b$$

But the combination of the boundaries of the two triangular systems $\partial(T_1 + T_2)$ is the same as the boundary $\partial(T_x)$ of the 2D planar system. Therefore,

$$\partial(T_1 + T_2) = \partial(T_x) = \partial_b + C^{b-1} - \tau_b + \tau_b - \partial_{b+1} - C^b \;\Rightarrow\; \partial(T_x) = \partial_b + C^{b-1} - \partial_{b+1} - C^b$$

Thus, $\partial(T_x) = \left(\partial_b + C^{b-1}\right) - \left(\partial_{b+1} + C^b\right)$. But from Figure 1.0 we realised that the composites of $\left(\partial_b + C^{b-1}\right) = C^{b-1}\partial_b$ and $\left(\partial_{b+1} + C^b\right) = \partial_{b+1}C^b$. However, since the closed system T_x is bounded, its boundary $\partial(T_x)$ is zero. Thus, $\partial(T_x) = 0$ which implies that:

$0=\left(\partial_b+C^{b-1}\right)-\left(\partial_{b+1}+C^b\right)=\left(C^{b-1}\partial_b\right)-\left(\partial_{b+1}C^b\right)$ Hence $C^{b-1}\partial_b=\partial_{b+1}C^b$ as required by the proposition.

2. The Extension of System

With the 2D planar system T_x, which consist of chain complexes and cochain complexes, we can form the sequence of 2D planar system. The natural homomorphism $F\left(b,x\right):\mathrm{C}^{b-1}\to C^b$ or $F\left(b,x\right):\mathrm{C}^{b}\to C^{b+1}$ for $b\geq 0$ provides each state *x* of T_x. For $b=0$, the homomorphism $F\left(0,x\right):\mathrm{C}^{0}\to C^1$ provides only one and initial state of 2D planar system such that the chain and cochain complexes shown at Figure 2 commutes

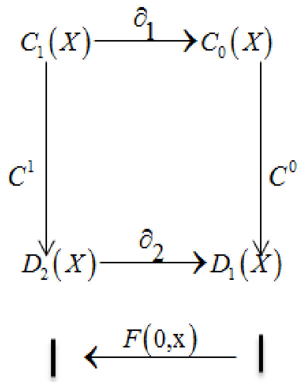

Figure 2: The First State of 2D Planar System

Similarly, when $b=1$, the extension of the system given by the homomorphism $F\left(1,x\right):\mathrm{C}^{1}\to C^2$ has two 2D planar systems, thus one extra 2D planar system from the initial one as shown in Figure 3.

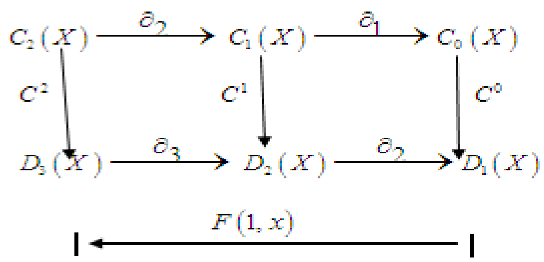

Figure 3: The Extension of 2D planar system

In each state, the closed surface area is the cocycle. Therefore, the initial state has one cocycle while the second state referred to as the first extended state has two cocycles. The extension of the system is the combination of 2-cells which means that each extended part of the system consists of a number of cocycles. Thus if $F\left(0,x\right)=1$ cocycle, $F\left(1,x\right)=2$, and $F\left(2,x\right)=3$ etc. The initial system is based on the notion of the chain map $\partial_1 : C_1 \to C_0$ and the cochain map $\mathrm{F}(0,\mathrm{x}) : C^0 \to C^1$. The addition of a new cochain group is dynamically determined by the homomorphism $F(b,x)$ from the system's initial state.

3. 1^{st} – Cohomology group H^1 and Extensions

Considering $F(b,x) : \mathrm{C}^b \to C^{b+1}$, for $b \geq 0$ the extension of the system in terms of the cochain complex is given as

$$\ldots \xleftarrow{F(3,x)} C^3 \xleftarrow{F(2,x)} C^2 \xleftarrow{F(1,x)} C^1 \xleftarrow{F(0,x)} C^0 \longleftarrow 0$$

In the form of 2D planar system the complete extended system E_n for $n = 1, 2 \ldots$ is illustrated at Figure 4.

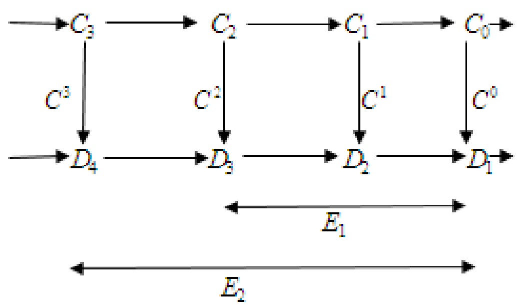

Figure 4: Extended System

If E_n is the extension of the system at each state for each value $n \in Z^+ - \{0\}$, then from Fig. 1.3, we have $E_n = F(0,\mathrm{x}) \bigcup_{b=1}^{n} F(b,x)$ where $F(0,x)$ and $F(b,x)$ are the inclusions in E_n. Each $E_n = F(0,\mathrm{x}) \bigcup_{b=1}^{n} F(b,x)$ is a complete bounded closed system consisting of cocycles and its respective coboundaries. Given that $KerF(\mathrm{b},x)$ is the set of cocycles or closed surface area of the cochain map C^b, its coboundary formed by 1-cells is the image of $KerF(\mathrm{b},x)$ denoted by $\mathrm{Im}\, of\ F(b,x)$ where the $\mathrm{Im}\, of\ F(b,x) \subset KerF(\mathrm{b},x)$. Since each extension E_n provides the same 2D planar closed systems with different sizes, we can compute the *1st* - cohomology group $H^1(E_n)$ of system at each state. The *1st*-cohomology group $H^1(E_n)$ is the ratio of cocycle to the coboundary. Thus, the 1^{st}–cohomology group of each extended system is given by:

$$H^1\left(E_n\right)=\frac{KerF\left(\mathrm{b},x\right)}{\mathrm{Im}\,of\ F\left(b,x\right)}$$

for $n=1,2,...$ and $b\geq 1$. Indeed, for $n=2,3,...$ the system increases in size but the shape is maintained. Each state of 2D planar closed systems has a complete cocycle as explained earlier. Thus, the first enlarged 2D planar closed systems $F(1,x)$ is obtained by adding $F(0,x)$ to $F(1,x)$ to obtain one complete cocycle as shown in Fig. 1.2. Similarly, the second 2D planar closed system is obtained by the combination of $F(0,x)$, $F(1,x)$ and $F(2,x)$. Since each enlarged 2D planar closed systems defines cocycle and every cocycle is bounded by coboundary which is the image of $KerF\left(\mathrm{b},x\right)$, the image of $KerF\left(\mathrm{b},x\right)$ is a divisor of the order of $KerF\left(\mathrm{b},x\right)$. Thus the 1st –cohomology group of each 2D planar closed system, which is defined by the ratio of cocycles to the coboundary, provides a definite value, which we shall consider as Z, the group of integers. The computed value of the 1st–cohomology group of each 2D planar closed system therefore shows all the integer combination of the 2D planar closed systems. An indication that there is 1, 2…increase in continuity and connectivity of the parts of the 2D planar closed systems. This is another technique of recovering information about the continuity and connectivity of complete 2D planar closed systems through the actions of its components. The algebraic invariant, thus the 1st– cohomology group H^1 provides the computable quantity for identifying the continuity and connectivity of the system. This is a new direction considered in studying the extensions of planar systems using cohomology group as the notion of continuous maps. The paper believes that this work can be enriched in many directions.

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