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The Interaction between Technical Change and Capital Investment Growth and Stability of R&D Model

Hanen Teka[a]*, Hatem Mhenni[b]

[a]*FSEGT, University of Tunis El Manar, Research laboratory ThEMA, Tunisia Lecturer, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia*

[b]*ESCT– University of Manouba, Research laboratory ThEMA, Tunisia*

[a]*Email: hkteka@iau.edu.sa*

[b]*Email: Hatem.Mhenni@essect.rnu.tn*

Abstract

Faced with an ever-changing economic environment, we have focused our attention on the study of the evolutionary relationship between the main factors of production. The originality of this work comes from a qualitative and quantitative analysis of the R&D model and considers the dynamic between labor, capital investment and technological progress. To understand the dynamics of the relationship between these variables in detail and to understand their long-term behavior, we have used the classical properties of Lotka Volterra's differential equations. The results of the Lyapunov function which we introduced have proved that the growth rate of technological progress and capital accumulation reach a stable long-term equilibrium. A numerical application was carried out on Canon industry and proved an evolution in the nature of the link between capital and R&D investment. We have attributed this behavioral change to the process of learning through interaction.

Keywords: R&D model; Lotka-Volterra system; Lyapunov function; Equilibrium analysis.

1. Introduction

R&D investment strategies are, by nature, dependent on many factors which often makes them extremely complex. The impact of these strategies is not known in advance and is difficult to predict which makes technological evolution a very difficult phenomenon to model.

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* Corresponding author.

In this work, we will use the differential equation system to study this type of evolution using an analysis of the link existing between the growth rate of the main factors of production[1]. Most previous studies have focused on the interdependence between the various factors of production [1, 2, 3, 4] without considering the nature of their mutual dependence. Our study is rare in that it examines this dependence using the obtained but not used Lotka Volterra model and analyses the competitive and cooperative relationship between the factors of production. It should be noted that several earlier works simply used the Lotka Volterra model to study the interdependence between two or more economic variables [5, 6].The classical properties of differential equation systems such as Lotka Volterra will enable us to prove the existence and uniqueness of a state of equilibrium. We have also used Lyapunov's workings to determine the stability of the long-term equilibrium between technological progress and capital accumulation. This paper will proceed in the following manner: In the course of section 2, we will explain the stages which led to the R&D model (such as Lotka Volterra). Throughout the third section we will carry out an equilibrium study that will highlight the coexistence of capital accumulation and technological progress. We shall furthermore construct a Lyapunov function which will help us in showing the asymptotic convergence towards a stable state of equilibrium.

A numerical application will then be carried out in order to verify the robustness of the model as well as the nature of the link between investment into capital and R&D. Section 4 illustrates the method of estimating the model parameters in order to determine the estimated values and also explains the interdependent nature between the economic variables in the case of the real market. The fifth section presents our conclusions. At the end we'll present our study's limitations.

2. Description of the R&D model

Following the example of many economic models, we consider an economy to be made up of two sectors: a goods sector which produces output and an R&D[2] sector which concerns itself with the production of knowledge. We will use a Cobb Douglas production function. Thus, the output Y(t) is produced at time t using the following equation:

$$Y(t) = ((1 - a_K)K(t))^{\alpha} \; (A(t)(l - a_L)L(t))^{\alpha'} \qquad (1)$$

where $\alpha \in (0, 1)$, represents the output elasticity of capital, $\alpha' = 1 - \alpha$ is the output elasticity of labor. The proportion a_L of the labor factor is used in the knowledge production section (R&D) while the proportion $1 - a_L$ is used in the production of goods sector. Moreover, the capital stock proportion a_K is used in the R&D sector while the proportion $1 - a_K$ is used in the production of goods sector. We furthermore assume that both sectors use the entirety of the knowledge stock[3] A(t).

In order to reduce text, we use $C_y = (1 - a_K)^{\alpha}(1 - a_L)^{\alpha'}$. The equation (1) takes the following form:

[1] Capital and Labour.

[2] Note that new knowledge and ideas are generated by the R&D sector.

[3] Both sectors use the entire stock of knowledge A as the use of an idea or knowledge in one place does not prevent its use elsewhere. Therefore, there is no need to divide the stock of knowledge between the two sectors of activity.

$Y(t) = C_Y(A(t))^{\alpha'} (K(t))^{\alpha} (L(t))^{\alpha'}$. The knowledge production function[4] presents itself as follows:

$$\dot{A}(t) = E\, (a_K K(t))^{\xi} (a_L L(t))^{\eta} (A(t))^{1+\theta} \qquad (2)$$

Where E is a strictly positive factor measuring the performance of the R&D sector. ξ, η are also positive factors. $\dot{A}(t)$ describes the flow of knowledge at time t. This function depends on the number of researchers, the existing stock of knowledge and of capital. In applying $C_K = E\, a_K^{\xi}\, a_L^{\eta}$, the function of knowledge production takes the following form: $\dot{A}(t) = C_K(K(t))^{\xi} (L(t))^{\eta}(A(t))^{1+\theta}$.

The study which we are carrying out on this work on the nature of the link between corporate strategies in terms of investment in R&D and capital is essentially based on the following system of production functions of goods and knowledge:

$$S_1 \begin{cases} Y(t) = C_Y (A(t))^{\alpha'} (K(t))^{\alpha} (L(t))^{\alpha'} \\ \dot{A}(t) = C_K (K(t))^{\xi} (L(t))^{\eta} (A(t))^{1+\theta} \end{cases}$$

We retain the law of capital accumulation used by [7], $\dot{K}(t) = s\, F(K(t), L(t)) - \delta K(t)$. Where s $\in$ [0,1] is the savings rate and δ is the capital depreciation rate.

Based on the knowledge production function $\dot{A}(t)$ and the capital accumulation law, we are studying the nature of the dynamics of the link existing between the knowledge stock growth rate $x(t) = \frac{\dot{A}(t)}{A(t)}$ and the capital growth rate $y(t) = \frac{\dot{K}(t)}{K(t)} = \frac{sY(t) - \delta K(t)}{K(t)}$. Unlike most works, we assume that the labor force grows according to $n(t) = \frac{\dot{L}(t)}{L(t)} > 0$ which converges to the term $n_\infty \geq 0$ when t tends towards $+\infty$. We also retain the fact that $n(t) - n_\infty$ verifies the following hypothesis : $\int_0^{+\infty} (n(t) - n_\infty)^2\, dt$ converge.

By replacing the knowledge stock growth rate with its value in (2), we obtain the following outcome:

$$x(t) = \frac{\dot{A}(t)}{A(t)} = C_K (K(t))^{\xi} (L(t))^{\eta} (A(t))^{\theta} \qquad (3)$$

We draw from this, the algorithm,

$$\ln(x(t)) = \ln(C_K) + \xi \ln(K(t)) + \eta \ln(L(t)) + \theta \ln(A(t)) \qquad (4)$$

[4] Similarly to [27] and [28] we are using the Cobb Douglas specification for the function of knowledge production.

A derivative with respect to the time of the relationship (4) allows us to write:

$$\frac{d}{dt}\left[ln(x(t))\right] = \frac{\dot{x}(t)}{x(t)} = \xi\,\frac{\dot{K}(t)}{K(t)} + \eta\,\frac{\dot{L}(t)}{L(t)} + \theta\,\frac{\dot{A}(t)}{A(t)}$$

Thus, the variation of the knowledge growth rate takes the following form:

$$\dot{x}(t) = x(t)\left[\xi\, y(t) + \theta\, x(t) + \eta\, n(t)\right] \qquad (5)$$

Proceeding in the same manner as before, we obtain the equation which represents the variation of the capital stock growth rate:

$$\dot{y}(t) = (y(t) + \delta)\left(\alpha'\left[x(t) - y(t) + n(t)\right]\right) \qquad (6)$$

The equations (5) and (6) enable us to therefore acquire the following system:

$$S_2 \begin{cases} \dot{x}(t) = x(t)\left[\theta\, x(t) + \xi\, y(t) + \eta\, n(t)\right] \\ \dot{y}(t) = \alpha'(y(t) + \delta)\left[x(t) - y(t) + n(t)\right] \end{cases}$$

The interaction of the three economic variables: the capital accumulation growth rate, the technological progress growth rate and the labor investment growth rate is thus expressed by the system S_2. We ascertain that this rule represents a modified version of the Lotka Volterra's prey-predator model[5] which constitutes a relatively simple description of the interaction between two species or populations [8, 9, 10] .

After its use in mathematics and biology, this type of equation system has been very successful in economics and management since it is considered a simple model for the treatment of fairly complex phenomena such as technological substitution [11] or organizational evolution [12]. With the help of the Lotka Volterra equation rules, the authors in [13] analyzed the competitive dynamic relationship of the Korean stock market and confirmed that competitive roles change with time. In [14] the author explored the effects of convergence between websites and the manner in which they affect the market. Reference [15] explored the Korean mobile telephone market. [16] studied the dynamics of durable consumer goods. Reference [17] explored economic growth through infectious diseases.

The objective of our study is to describe, using a modified version of Lotka Volterra model (S_2 system), the nature of interaction between key economic variables of firm evolution. As $\frac{\dot{x}(t)}{x(t)}$ and $\frac{\dot{y}(t)}{y(t)}$ depend linearly to x and y, we can distinguish the nature of the evolution of these variables using the indicated parameters of the S_2

[5] The equations of Lotka Volterra which we refer to using the term prey predator model are a couple of non-linear differential equations of the first order and are generally used to describe the dynamics of biological systems in which a predator and its prey interact. They were proposed independently by the authors in [8] and [29].

system. In effect, the sign of parameters ξ, θ and α' represent an improvement in growth (if positive), indifference (if null) or the effect of suppression (if negative) of a variable on itself or on another[6]. $x^2(t)$ and $y^2(t)$ represent the terms of self-action which generally show a negative feedback as the parameters θ and $-\alpha'$ simply describe the interaction of a variable on itself. Thus ξ refers to the effect of an increase in variable y on variable x and α' represents the effect of an increase of variable x on variable y.

In what follows, we will use certain classical properties of the S_2 system to highlight the existence and the singularity of the global solution (x(t), y(t)) of the model, and will study its asymptotic behaviour[7]. This qualitative study will follow a numerical application in order to estimate the parameters of the model and determine, as a result, the nature of interaction which exists between the studied variables.

3. Analysis of the R&D model

Since investment in R&D and in capital are both non-negative variables we must restrict our attention to the side of the positive quadrant plane where for any initial solution we have: $\{(x_0, y_0)/x_0 \geq 0, y_0 \geq 0\} \subset R^2$.

3.1 Coexistence of the "predator-prey" solution

The analysis of the competitive relationship between economic variables through the use of the Lotka Volterra R&D model (S_2) gives us an idea of the equilibrium state and also allows us to illustrate the trajectory over time. Thus, taking into account the S_2 system and the instance whereby isoclines $\dot{x} = 0$ and $\dot{y} = 0$, intersect, we can prove the existence of an equilibrium point (x_∞, y_∞), i.e. the coexistence of variables representing the technological progress and the capital accumulation. We subsequently introduce a Lyapunov[8] function to S_2 and use it to prove the global stability of the model.

The principle goal of this section is to determine possible solutions for S_2. As it is impossible to calculate these in an analytical way (quantitative), we shall therefore focus on a rather more qualitative study of the S_2 system.

Our work aims to show that if $detA > 0$, and if $x(0) > 0$, $y(0) > -\delta$ the S_2 system of equations allows a global solution which confirms that $x(t) > 0$, $y(t) > -\delta$ whenever $t \geq 0$. Proof of this relationship involves the study of local existence and the singularity of the solution by using theorem 1 as shown by [5][9].

3.1.1 Local existence and singularity

In this paragraph, we will use one of the classic outcomes regarding the existence and singularity of differential

[6] [30].

[7] As the dynamic process is described by differential equations in which the independent variable is time, the theory intends to predict the ultimate behaviour of the solutions to the equation in the distant future (when $t \to \infty$). It therefore involves understanding, from an initial situation, towards which values the growth rate of knowledge stock and capital stock will evolve.

[8] One general way to establish the overall stability of a differential equation system is to use Lyapunov's function. See [31, 34, 33].

[9] Page 6.

equation solutions[10], which involves a given initial value (x_0, y_0), and uses the Cauchy-Lipschitz theorem[11].

Recall that if f, g : $R^3 \rightarrow R$ of class C^1 such as:

$$(1)\begin{cases} \dot{x}(t) = f(x,y,t) \\ \dot{y}(t) = g(x,y,t) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

Then, there exist $t_{max} > 0$, so long as the problem (1) allows a unique solution in $[0,t_{max})$. Then we show that if $(x(t),y(t))$ is bounded on $[0, t_{max})$ then $t_{max} = +\infty$, that is to say the solution which exists is global.

The application of this outcome to our study whereby $f(x, y, t) = x(t)\,[\theta\, x(t) + \xi\, y(t) + \eta\, n(t)]$ and $g(x, y, t) = \alpha'(y(t) + \delta)[x(t) - y(t) + n(t)]$, allows us to state that $t_{max} > 0$, exists here and as such S_2 permits a unique solution for $[0,t_{max})$. In the following paragraph, we'll prove that the solution $(x(t),y(t))$ is bounded on $[0, t_{max})$, which allows us to have $t_{max} = +\infty$, which implies the solution is global.

3.1.2 Global Attractivity

Before revealing the global solution of S_2, it is first necessary to prove the stability of the rule $S_\delta = \{(x, y)/\, x > 0, y > -\delta\}$, that is to say that if there exists a solution based on time $[0, t_{max})$ and if $x(0) > 0$ and $y(0) > -\delta$ then $x(t) > 0$, $y(t) > -\delta$ on $[0,t_{max})$. Proof can be found in the Appendix.

We will now try to show that the solution which exists is global, and $t_{max} = +\infty$. It is sufficient for us to prove that $(x(t), y(t))$ is bounded, and for that we will use Lyapunov's theorem [18]. The construction of the Lyapunov function requires knowledge of the equilibrium state in the dynamic system S_2 to thus obtain the point (x_∞, y_∞), where neither of the two variables evolves.

We start by writing the S_2 system in its matrix form: $\dot{X}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} x(t)\left[\xi\, y(t) + \theta\, x(t) + \eta\, n(t)\right] \\ \alpha'(y(t) + \delta)\left[x(t) - y(t) + n(t)\right] \end{pmatrix}$, Which is equivalent to, $\dot{X}(t) = diag(x(t), y(t) + \delta)[AX(t) + b]$ Where, $A = \begin{pmatrix} \theta & \xi \\ \alpha' & -\alpha' \end{pmatrix}$, $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $b = \begin{pmatrix} \eta n(t) \\ \alpha' n(t) \end{pmatrix}$.

The determinant of matrix A is thus equal to $\alpha'(\theta + \xi)$, which enables us to say that the point representing the equilibrium state between capital and knowledge growth rates $(x_\infty, y_\infty) \in R_+^2$, as we explain below, is the unique point of equilibrium if, and only if, $detA \neq 0$. As the inverse matrix is given by:

[10] Like S_2.

[11] This theorem was first introduced by Cauchy in 1844 then generalized by Lipschitz; not long after, a demonstration of this fundamental outcome could be found in [34], chapter 15.

$$A^{-1} = \frac{1}{-\alpha'(\theta+\xi)}\begin{pmatrix} -\alpha' & -\xi \\ -\alpha' & \theta \end{pmatrix}$$

The equilibrium point(s) are written as follows:

$$X_\infty = A^{-1}(-b_\infty) = \frac{1}{-\alpha'(\theta+\xi)}\begin{pmatrix} -\alpha' & -\xi \\ -\alpha' & \theta \end{pmatrix}\begin{pmatrix} -\eta n_\infty \\ -\alpha' n_\infty \end{pmatrix} = \frac{1}{(\theta+\xi)}\begin{pmatrix} -(\eta+\xi)n_\infty \\ (\theta-\eta)n_\infty \end{pmatrix}$$

Hence, $\begin{cases} x_\infty = -\dfrac{\eta+\xi}{\theta+\xi}n_\infty \\ y_\infty = \dfrac{\theta-\eta}{\theta+\xi}n_\infty \end{cases}$

We find that at equilibrium, the values of the two system variables, depending on the four parameters η, ξ, θ, n_∞ remain stable indefinitely[12], We also remark that the capital depreciation rate "δ" has no impact on the equilibrium point.

We will now proceed to the second stage of our study which consists of determining the Lyapunov[13] function "*V(x(t),y(t))*" which has certain properties linked to system (1).

Based on the literature[14] surrounding differential equation systems[15], where the Lyapunov function is conventional for n(t) = 0, we construct a Lyapunov function in a general context where the employment growth rate n(t) is a variable which depends on time; this represents a novelty compared with previous works.

Let,

$$V(x(t),y(t),t) = x(t) - x_\infty - x_\infty ln\frac{x(t)}{x_\infty} - \frac{2\theta+\xi}{\alpha'}\left(y(t) - y_\infty - (y_\infty+\delta)ln\frac{y(t)+\delta}{y_\infty+\delta}\right) - A\int_0^t (n(s)-n_\infty)^2 ds \qquad (7)$$

where A is a positive constant which we define later on. It should be noted that the function *V(x(t), y(t), t)* is clearly defined in $[0, t_{max})$, because for any t ∈ $[0, t_{max})$, the solution of the differential equations system (S_2) verifies: $x(t) > 0$ and $y(t) > -\delta$.

The derivative of the equation (7) in relation to time gives us the following outcome:

$$\frac{d}{dt}V(x(t),y(t),t) = \left(1 - \frac{x_\infty}{x(t)}\right)\dot{x}(t) - \frac{2\theta+\xi}{\alpha'}\left(1 - \frac{y_\infty+\delta}{y(t)+\delta}\right)\dot{y}(t) - A(n(t)-n_\infty)^2$$

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[12] Not dependent on time.

[13] In effect, the Lyapunov function is a type of dissipating energy which serves as a brake to the solution so that it does not explode.

[14] [32, 34, 36, 38].

[15] Lotka- Volterra systems.

By replacing $\dot{x}(t)$ and $\dot{y}(t)$ with their values in (S_2), we obtain the following equation:

$$\frac{d}{dt}V(x(t),y(t),t) = \tilde{x}(t)\left(\theta\, x(t)+\xi\, y(t)+\eta\, n(t)\right) - \left(2\theta+\xi\right)\tilde{y}(t)\left(x(t)-y(t)+n(t)\right) - A(\tilde{n}(t))^2$$

With $\tilde{x}(t) = x(t) - x_\infty$; $\tilde{y}(t) = y(t) - y_\infty$; $\tilde{n}(t) = n(t) - n_\infty$.

By using the fact that $\theta\, x_\infty + \xi\, y_\infty + \eta\, n_\infty = 0$ and $x_\infty - y_\infty + n_\infty = 0$, the derivative of the function $V(x(t), y(t), t)$ takes the following form: $\frac{d}{dt}V(x(t),y(t),t) = \tilde{x}(t)\left(\theta\, \tilde{x}(t)+\xi\, \tilde{y}(t)+\eta\, \tilde{n}(t)\right) - \left(2\theta+\xi\right)\tilde{y}(t)\left(\tilde{x}(t)-\tilde{y}(t)+\tilde{n}(t)\right) - A(\tilde{n}(t))^2$.

A simple calculation allows us to say that:

$$\frac{d}{dt}V(x(t),y(t),t) = \theta\left(\tilde{x}(t)-\tilde{y}(t)\right)^2 + \left(\theta+\xi\right)\left(\tilde{y}(t)\right)^2 + \eta\, \tilde{n}(t)\left(\tilde{x}(t)-\tilde{y}(t)\right) - \left(2\,\theta+\xi-\eta\right)\tilde{n}(t)\,\tilde{y}(t) - A\left(\tilde{n}(t)\right)^2 \tag{8}$$

As $$\eta\, \tilde{n}(t)\left(\tilde{x}(t)-\tilde{y}(t)\right) \leq -\frac{\theta}{2}\left(\tilde{x}(t)-\tilde{y}(t)\right)^2 + \frac{\eta^2}{-2\,\theta}\left(\tilde{n}(t)\right)^2 \tag{9}$$

and

$$-(2\,\theta+\xi-\eta)\,\tilde{n}(t)\,\tilde{y}(t) \leq -\frac{\theta+\xi}{2}\left(\tilde{y}(t)\right)^2 - \frac{\left(2\,\theta+\xi+\eta\right)^2}{\theta+\xi}\left(\tilde{n}(t)\right)^2 \tag{10}$$

We can then combine (8), (9) and (10), in order to obtain the following inequality:

$$\frac{d}{dt}V(x(t),y(t),t) \leq \frac{\theta}{2}\left(\tilde{x}(t)-\tilde{y}(t)\right)^2 + \frac{\theta+\xi}{2}\left(\tilde{y}(t)\right)^2 + \left(\frac{\eta^2}{-2\,\theta} + \frac{\left(2\,\theta+\xi+\eta\right)^2}{-(\theta+\xi)} - A\right)\left(\tilde{n}(t)\right)^2 \tag{11}$$

If $A = \frac{\eta^2}{-2\theta} + \frac{\left(2\theta+\xi+\eta\right)^2}{-(\theta+\xi)} > 0$, then (11) is written as follows:

$$\frac{d}{dt}V(x(t),y(t),t) \leq \frac{\theta}{2}\left(\tilde{x}(t)-\tilde{y}(t)\right)^2 + \frac{\theta+\xi}{2}\left(\tilde{y}(t)\right)^2 \tag{12}$$

This allows us to state that function V is declining. Moreover, we have:

$V(x(t), y(t), t) \leq V(x(0), y(0), 0) = V_0, \quad \forall t \in [0, t_{max})$.

Using the notion that $\int_0^{+\infty}\left(\tilde{n}(t)\right)^2 dt$ converges, we obtain for any $t \in [0, t_{max})$.

$$\tilde{x}(t) - x_{\infty}\, ln\frac{x(t)}{x_{\infty}} - \frac{2\theta+\xi}{\alpha'}\left(\tilde{y}(t) - (y_{\infty}+\delta) ln\frac{y(t)+\delta}{y_{\infty}+\delta}\right) \le A\int_0^{+\infty}(\tilde{n}(s)^2)ds + V_0 \tag{13}$$

As $x_\infty \geq 0$ and $y_\infty \geq 0$ thus

$$x(t) - x_{\infty}\, ln\frac{x(t)}{x_{\infty}} \ge 0 \quad \text{if} \quad x(t) \ge 0 \tag{14}$$

And

$$y(t) - y_{\infty} - \left(y_{\infty}+\delta\right) ln\frac{y(t)+\delta}{y_{\infty}+\delta} \ge 0 \quad if \quad y(t) \ge -\delta \tag{15}$$

By combining $-\frac{2\theta+\xi}{\alpha'} > 0$ with the inequalities (13), (14) and (15) we obtain for any $t \in [0,t_{max})$

$$0 \le \tilde{x}(t) - x_{\infty}\, ln\frac{x(t)}{x_{\infty}} \le A\int_0^{+\infty}(\tilde{n}(s)^2)ds + V_0 \tag{16}$$

and

$$0 \le \tilde{y}(t) - (y_{\infty}+\delta) ln\frac{y(t)+\delta}{y_{\infty}+\delta} \le \frac{-\alpha'}{2\theta+\xi}\left(A\int_0^{+\infty}(\tilde{n}(s)^2)ds + V_0\right) \tag{17}$$

With the relationship (16), we deduce that $x(t)$ is bounded on $[0, t_{max})$. Similarly, the relationship (17) proves that $y(t)$ is bounded on $[0, t_{max})$. Thus, we conclude that the vector $(x(t),y(t))$ is bounded on $[0, t_{max})$. And as theorem 1 indicates, $t_{max} = +\infty$, the solution $(x(t),y(t))$ is thus global.

After having proved the general existence and singularity of the model's solution $(x(t),y(t))$, we continue our qualitative study by analyzing the convergence of $(x(t),y(t))$ to observe its long-term behavior.

3.2 Convergence towards an equilibrium state

This section is based on studying the convergence of the (S_2) global solution towards an equilibrium state (x_∞, y_∞). We will proceed in two stages: the first consists of proving the solutions convergence whilst the second is dedicated to showing that the convergence does lead towards the equilibrium point (x_∞, y_∞).

3.2.1 Convergence of the model's solution

The key element which will allow us to show the convergence of the solution $(x(t), y(t))$ of (S_2) consists of using the function $V(x(t), y(t), t)$ defined in (7) and also introduces a second Lyapunov function $W(x(t), y(t), t)$, that is close to the first in order to guarantee that the coefficients of $W(x(t), y(t), t)$ remain close to those of $V(x(t),y(t),$

$t)$, in a manner which preserves the decline of the second function.

Let, for any $t \geq 0$

$$W(x(t),y(t),t) = \tilde{x}(t) - x_\infty \ln \frac{x(t)}{x_\infty} - \frac{2\theta+\xi+\sqrt{2\theta(\xi+\theta)}}{\alpha'}\left(\tilde{y}(t) - (y_\infty+\delta)\ln\frac{y(t)+\delta}{y_\infty+\delta}\right) - B\int_0^{+\infty}(\tilde{n}(s)^2)ds, \tag{18}$$

It should be noted that $-\frac{2\theta+\xi+\sqrt{2\theta(\xi+\theta)}}{\alpha'} > 0$ and B, is a positive constant which will be determined later on.

Proceeding in the same way as before, we will take the derivative of the function W with respect to time which gives us:

$$\frac{d}{dt}W(x(t),y(t),t) = \theta\left(\tilde{x}(t) - (1-\sqrt{\frac{\xi+\theta}{2\theta}})\tilde{y}(t)\right)^2 + \frac{\theta+\xi}{2}\left(\tilde{y}(t)\right)^2 + \theta\left(\tilde{x}(t) - (1-\sqrt{\frac{\xi+\theta}{2\theta}})\tilde{y}(t)\right)\tilde{n}(t)$$

$$-(\theta+\xi+\sqrt{\frac{5\theta(\xi+\theta)}{2}})\tilde{y}(t)\,\tilde{n}(t) - B(\tilde{n}(t))^2 \tag{19}$$

We will use the mathematical relationship: $ab \leq \frac{a^2+b^2}{2}$ to be able to write the following inequalities:

$$\theta\left(\tilde{x}(t) - (1-\sqrt{\frac{\xi+\theta}{2\theta}})\tilde{y}(t)\right)\tilde{n}(t) \leq -\frac{\theta}{2}\left(\tilde{x}(t) - (1-\sqrt{\frac{\xi+\theta}{2\theta}})\tilde{y}(t)\right)^2 - \frac{\theta}{2}(\tilde{n}(t))^2 \tag{20}$$

and

$$-(\theta+\xi+\sqrt{\frac{5\theta(\xi+\theta)}{2}})\tilde{y}(t)\tilde{n}(t) \leq -\frac{\theta+\xi}{4}(\tilde{y}(t))^2 - \frac{\left(\theta+\xi+\sqrt{\frac{5\theta(\xi+\theta)}{2}}\right)^2}{\theta+\xi}(\tilde{n}(t))^2 \tag{21}$$

Using (19), (20), (21) and by giving to B the following value : $B = -\frac{\left(\theta+\xi+\sqrt{\frac{5\theta(\xi+\theta)}{2}}\right)^2}{\theta+\xi} - \frac{\theta}{2}$, we obtain the following inequality:

$$\frac{d}{dt}W(x(t),y(t),t) \leq \frac{\theta}{2}\left(\tilde{x}(t) - (1-\sqrt{\frac{\xi+\theta}{2\theta}})\tilde{y}(t)\right)^2 + \frac{\xi+\theta}{4}\left(\tilde{y}(t)\right)^2 \tag{22}$$

As $\theta < 0,\ (\theta+\xi) < 0$ thus $\frac{d}{dt}W(x(t),y(t),t) \le 0$ which implies the decline of the function *W(x(t),y(t), t)*. According to (14) and (15) we have for any $t \ge 0$, $V(x(t),y(t),t) \ge -A\int_0^{+\infty}(\tilde{n}(s)^2)ds$,and $W(x(t),y(t),t) \ge -B\int_0^{+\infty}(\tilde{n}(s)^2 ds$. Since *V(x(t),y(t),t)* and *W(x(t),y(t),t)* are decreasing then they are converging. Furthermore, by using the expressions of *V(x(t),y(t))* and *W(x(t),y(t))* we obtain the following result:

$$\frac{\sqrt{2\theta(\xi+\theta)}}{\alpha'}(y(t)-y_\infty-(y_\infty+\delta)ln\frac{y(t)+\delta}{y_\infty+\delta}=V(x(t),y(t),t)-W(x(t),y(t),t)\ +(B-A)\int_0^t(\tilde{n}(s)^2)ds. \tag{23}$$

It becomes clear, then, that the term to the right of the equation (23) converges. In effect, as the linear combination of the two converging functions *V(x(t),y(t),t)* and *W(x(t),y(t),t)* is converging and $(B-A)\int_0^t(\tilde{n}(s)^2)ds$ converges towards $(B-A)\int_0^{+\infty}(\tilde{n}(s)^2)ds$, which implies the convergence of $y(t)-y_\infty-(y_\infty+\delta)ln\frac{y(t)+\delta}{y_\infty+\delta}$. In conclusion, the growth rate of capital investment, *y(t)*, converges.

Similarly, using the fact that *V(x(t), y(t), t)* converges as does *y(t)*, we deduce that $x(t)-x_\infty-x_\infty\,ln\frac{x(t)}{x_\infty}$ converges, which enables us to say that *x(t)* is also converging.

After having proved the convergence of the capital accumulation and technological progress growth rates *(x(t),y(t))*, we will confirm its convergence to the equilibrium point (x_∞, y_∞).

3.2.2 Convergence of the model's solution to the equilibrium

The second half of this qualitative analysis aims to confirm the S_2 solution's convergence to the point of equilibrium (x_∞, y_∞). In effect, integrating the relationship (12) between 0 and t allows us to obtain the following result:

$-\frac{\theta}{2}\int_0^t(\tilde{x}(s)-\tilde{y}(s))^2ds-\frac{\theta+\xi}{2}\int_0^t(\tilde{y}(s))^2\,ds \le V(x(0),y(0),0)-V(x(t),y(t),t).$ As $V(x(t),y(t),t) \ge -A\int_0^{+\infty}(\tilde{n}(s)^2)ds$, thus

$$-\frac{\theta}{2}\int_0^t(\tilde{x}(s)-\tilde{y}(s))^2ds-\frac{\theta+\xi}{2}\int_0^t(\tilde{y}(s))^2\,ds \le V(x(0),y(0),0)+A\int_0^{+\infty}(\tilde{n}(s)^2)ds.$$

Recall that $-\theta > 0$ and $-(\theta+\xi) > 0$ thus,

$$\int_0^t (\tilde{x}(s) - \tilde{y}(s))^2 ds \leq \frac{2}{-\theta}\left(V(x(0), y(0), 0) + A\int_0^{+\infty} (\tilde{n}(s)^2\, ds)\right). \qquad (24)$$

And

$$\int_0^t (\tilde{y}(s))^2 ds \leq \frac{2}{-\theta - \xi}\left(V(x(0), y(0), 0) + A\int_0^{+\infty} (\tilde{n}(s))^2\, ds\right). \qquad (25)$$

As t tends towards infinity in (24) and (25) we deduce that $\int_0^{+\infty} (\tilde{x}(s) - \tilde{y}(s))^2 ds$ and $\int_0^{+\infty} (\tilde{y}(s))^2 ds$ do converge.

From what is above, we have already obtained the convergence of $x(t)$ and of $y(t)$, where we can state that $(\tilde{x}(t) - \tilde{y}(t))^2 ds$ and $(y(t))^2$ converges, and thus $(\tilde{x}(t) - \tilde{y}(t))^2$ and $(\tilde{y}(t))^2$ converges towards zero[16], that is to say the solution of the S_2 model, $(x(t), y(t))$, converge towards the equilibrium point (x_∞, y_∞).

We continue our study into the dynamics of the main factors of production growth rates by building on a numerical analysis of one of the most important optical industries, Canon.

4. Numerical application

As it is always important to verify the authenticity of any given theoretical study, in what follows we shall tackle an example from the real market, Canon industry.

4.1 Description of the practical case

Essentially, our study focuses on the S_2 model which allows us to describe the evolution of the nature of interaction between economic variables $x(t)$, $y(t)$ and $n(t)$.

Investments in capital and R&D are known as being variables which intervene in the same industrial environment and so there exists a classical way to describe the various relationships between $x(t)$ and $y(t)$, as illustrated in the table below:

Table 1: The nature of relationships between x(t) and y(t)

Signe de α’				
		-	**0**	+
Signe de ξ	-	competition		
	0	Amensalism	Neutralism	
	+	Predation	Commensalism	Mutualism

Source: [19]

[16] This result is known in the case of general integrals.

Based on the work of [20, 21, 22] and on the data in the above table, we can distinguish 6 modes of interaction which can exist between the R&D and capital growth rates *x(t)* and *y(t)*, and that depends on the sign of the model's parameters. This brings us to estimate the parameters of S_2 in order to know their signs and identify as a result the mode of interaction which really exists between the growth rate of both capital accumulation and technological progress in Canon, an industry with a high degree of innovation.

4.2 Parameters Estimation

Our studied S_2 system involves 5 parameters which we will try to estimate using experimental data from Canon industry in order to identify the nature of the relationship between variables.

4.2.1 Initial data

The numerical evaluation of the S_2 model is based on quarterly data describing the growth rates of investment in capital, R&D and labour in the Canon industry. These data have been compiled from annual reports that are available online from the first quarter of 2000 to the final quarter of 2014, covering a total of 56 quarters. Several tests with simulated data parameters were carried out to allow us to adopt an initial estimate of $(\alpha'_0, \theta_0, \xi_0, \eta_0, \delta_0)$. The latter is used to give a more precise estimate of the parameters of the model.

4.2.2 Model parameters estimation

In order to evaluate the estimated values of the model's parameters, we will use the information we have available on the evolution of time data of the variables *x(t)* and *y(t)*. We'll estimate the parameters of S_2 by adapting them to the experimental data. The estimation of the unknown parameters of the model (ξ, θ, η, α' and δ) are created by using the values of $\dot{x}$ and $\dot{y}$ which we obtained from experimental data. This will prove benefits when searching for parameters that adapt well to the experimental data. We thus define a function of the least squares which measures the difference between the simulated values ($\dot{x}$ and $\dot{y}$) and the experimental data. We shall find the minimum error starting from an initial state. This practical study was carried out on Matlab where the fminsearch algorithm was performed with random initial values in order to ensure the convergence of the algorithm towards a global minimum rather than being restricted by a local minimum. Table 2 below describes the estimated values of the model parameters.

Table 2: The parameters estimated values

Parameter	Estimated value
p(l)= ξ	0.9152
p(2)= θ	-3.2302
p(3)= η	1.8632
p(4) = α'	0.8668
p(5) = δ	0.7247

Application of Matlab (fminsearch and ODE 45) produced an S_2 system with the following estimated values:

$$S_2 \begin{cases} \dot{x} = x(t)\left[-3.2302\, x(t) + 0.9152\, y(t) + 1.8632\, n(t)\right] \\ \dot{y} = 0.8668\left(y(t) - 0.7246\right)\left[x(t) - y(t) + n(t)\right] \end{cases}$$

The estimated parameters of the S_2 system illustrate significant information regarding the dynamics of the described process. In fact, we found that the coefficients ξ and α' are of the same sign[17], which indicates that the results of the modified Lotka Volterra (S_2) simulation model reveal the nature of interaction between the growth rate of capital and R&D investment in showing a “mutual” competitive relationship according to the classification presented in Table 1. This type of link [20, 23] indicates that the R&D investment growth rate exerts a positive influence on the evolution of the capital investment growth rate and that the latter also exerts a positive influence on the evolution of the R&D investment growth rate. This type of interaction is thus a kind of intimate association with a reciprocal benefit. The capital investment growth rate can thus be described as a “bodyguard” of the of R&D investment growth rate. The two variables therefore describe a cyclical link that is win-win for both parties. In addition, the relative curves for the growth rates of investments in R&D and capital, which are based on the estimated functions of x and y are illustrated in Figure 1 below. These curves follow approximately the same trajectory as the real data. This signifies that the S_2 system perfectly explains the evolution of the growth rates of the investment in capital and R&D at Canon industry.

4.3 Analysis of the "technological progress-capital accumulation" relationship

As illustrated above, There are many kinds of relationships between capital accumulation and technological progress and are classified according to the sign of the parameters ξ and α'. Based on the results of the estimated values presented in Table 2, we find that the coefficients ξ and α' are positive, indicating thus a "mutual" relationship between the growth rate of the investment in capital and in R&D. However, figures 2 and 3 show that at the start of the study period, the growth rate of the capital and R&D investment interact following the Lotka Volterra predator-prey relationship. In fact, when the variable x tends to increase, y tends to decrease and vice versa. This phenomenon can be explained by the fact that the capital investment is key to the success of innovation projects and its growth ensures the evolution of R&D. In a sense, the evolution of capital ensures the survival of innovation projects within a firm which explains the predator-prey relationship between capital accumulation and technological progress. In some ways, R&D investment acts as a sort of predator whilst capital investment (prey) grows exponentially under the hypothesis that the population of prey can find enough" food" at any time and starts to decrease when a sufficient number of predators appear. Whereas, the growth rate of investment in R&D (predator) decreases in an exponential manner and grows when there is sufficient prey.

[17] ξ and α' are positive.

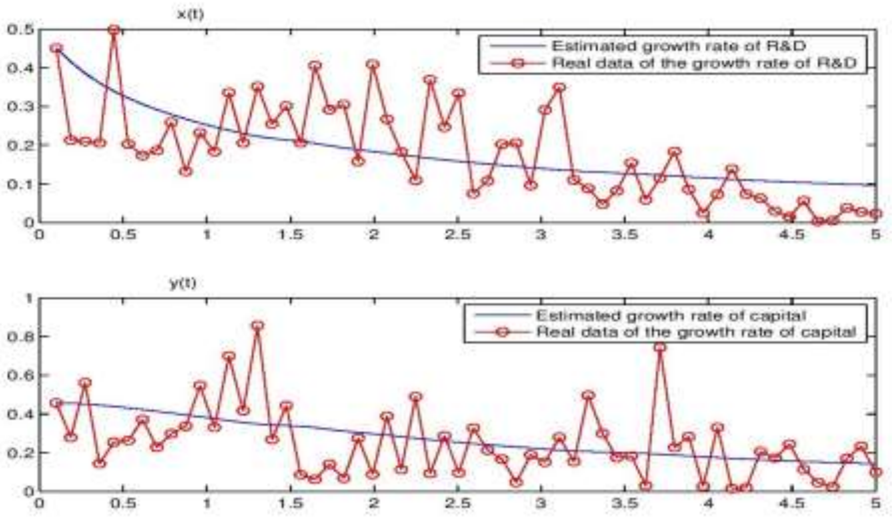

Figure 1: Real and simulated data

As time passes we see that, as indicated in figure 2, a cycle of increase and decrease repeats itself. In reality, this is a simple description of the Lotka Volterra model which is essentially used in biology[18] Then after a certain time[19], the two variables witness a mutation towards a new form of relationship[20] known under the name of mutualism. In effect, this kind of interaction between variables or species has been classified by specialists according to several categories; symbiosis, commensalism, cooperation, facilitation, reciprocal altruism and mutual aid [24].

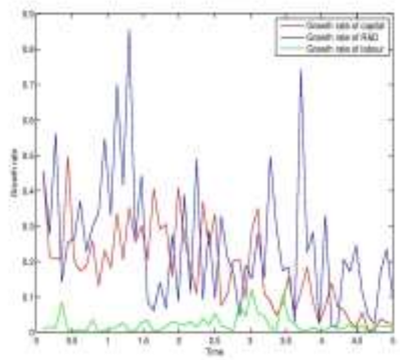

Figure 2: R&D and K growth rates as a LV Model

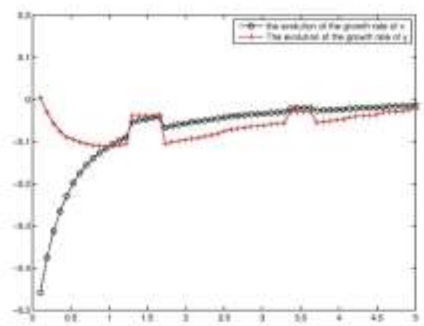

Figure 2: The Evolution of the growth rate of the Investment on capital and R&D

[18] [38].

[19] Intermediate and long periods.

[20] A type of mutation.

This mutualism has also been proved by the sign of the estimated parameters ξ and α' . Figure 3 thus describes the mutation of *x(t)* and *y(t)* towards a new type of relationship where the evolution of one is more beneficial to the evolution of the other before the global convergence towards the equilibrium.

4.4 Equilibrium analysis

The analysis of the S_2 model can provide us with a certain amount of information on the state of equilibrium and the change of the trajectory depending on time. This analysis furthermore allows us to identify the stability of the equilibrium. We begin this work by finding the equilibrium state (x_∞, y_∞) and analyzing the stability of the growth rate of capital investment and R&D investment of the S_2 system by setting their evolution to zero:

$$\left(\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \text{ and } n(t) \to n_\infty \right).$$

At the equilibrium point, the S_2 differential equations are equivalent to the following algebraic equations:

$$\begin{cases} \left[\theta x(t) + \xi y(t) + \eta n_\infty\right] = 0 \\ \alpha' \left[x(t) - y\delta(t) + n_\infty\right] = 0 \end{cases}$$

The system of equations above can also be expressed in the following manner:

$$x = \frac{\eta n_\infty - \xi y}{\theta}, \quad y = \frac{\alpha' n_\infty - \alpha' x}{-\alpha'} \tag{26}$$

The two expressions $\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 0$ grow to the equilibrium point where capital accumulation and technological progress coexist without a dynamic change.

Following the equations presented above, if $x(t) < \frac{\eta n(t) - \xi y(t)}{\theta}$ then $\frac{dx}{dt} > 0$ which signifies that the growth rate of the investment in R&D [21] will witness an increase. In the opposite case, when $x(t) > \frac{\eta n(t) - \xi y(t)}{\theta}$ and so $\frac{dx}{dt} < 0$, the growth rate of R&D investment sees a decrease. Following the same logic, the growth rate of capital investment will increase if $y(t) < \frac{\alpha' n(t) - \alpha' x(t)}{-\alpha'}$ and vice versa. In the long term, the growth rates of investment in capital and R&D reach the equilibrium at the point (x_∞, y_∞), which is the point of intersection of the two lines of the equation (26). The stability of this equilibrium point

21 Recall that x(t) represent the growth rate of R&D investment.

depends on the values of the S_2 coefficients.

Figure 3 describes the simulated trajectories of the growth rate of the investment in capital and in R&D in Canon industry. We can clearly see that at the start, the growth rate of one increases whilst the other decreases (following the predator-prey behavior), then the two variables witness a mutual interaction which leads to a convergence towards the point of equilibrium which is the point (0,0). This change in behavior (from predator-prey to mutualism can be explained by the effect of learning through interaction in the sense of B.A. Lundvall).

Moreover, using the estimated values of the model parameters, we can prove with the help of Lyapunov functions that investment in R&D and in capital converge towards a stable equilibrium point which is asymptotically stable as the given conditions of the parameters ($-\theta = 3.2302$ is > 0 as well as $-(\theta + \xi) = 4{,}1454$ is > 0) permit the following relationships:

$V(x(t),y(t),t) \geq 0$ and $\frac{dV(x(t),y(t),t)}{dt} \leq 0$, $W(x(t),y(t),t) \geq 0$ and $\frac{W(x(t),y(t),t)}{dt} \leq 0$.

The equilibrium relationships between capital accumulation and technological progress are stable in the long-term. The two variables thus reach a phase of maturity.

5. Conclusion

In this paper we have attempted to develop a modified version of the Lotka Volterra model in order to study the interactive dependence between the main factors of production. Specifically, how the relationship between technological progress and capital accumulation evolve over time.

To the best of our knowledge, very few studies focus specifically on the dynamics of the link that exists between technological progress and capital accumulation. The originality of our work therefore lay in the idea of illustrating and analyzing the evolution of the relationship between these economic variables[22] in the same industry[23].

The analytical results of our research illustrate a change in behaviour over time with regards to the evolution of two key economic variables (investment in R&D and in capital). In fact, we have recorded the passage from a predator-prey relationship to a more mutual relationship before reaching a stable long-term equilibrium.

We explain this type of behaviour mutation by the birth of a new type of learning through interaction in the sense of the authors in [25] which meant that technological progress and accumulation of capital learn to draw mutual profit from each other as time goes by. By basing ourselves on the numerical application of results obtained by Canon industry, we suggest the coexistence of technological progress and capital accumulation following two forms of relationship, competition and cooperation.

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[22] Growth rate of the investment in R&D, in capital and in labour.

[23] Canon Industry.

The labour growth rate also plays an important role in the area of our model. In effect, the growth rate of both technological progress and capital accumulation in the long run is controlled by the labour growth rate. In addition, the results of the Lyapunov investigation show the existence of a stable overall equilibrium. It should be noted that if the model solution is stable, S_2 converges to an equilibrium position which shows that the growth rates of technology and capital accumulation will converge to zero and the firm will stop growing and become stagnated.

6. Limitation of study

In this paper we developed a modified version of the Lotka-volterra model. The results of this study allowed us to have a better understanding of the interactive relationships among main economic variables. We have also shown the impact of the learning by interaction on the dynamic link between investment in R&D and capital investment.

The numerical method used in this study to find a global asymptotic solution didn't take into account possible chaotic system. Since, the R&D model is characterised by a high degree of uncertainty due to the irregular events in the market, we will try, in future studies to consider financial crises when adopting the Lotka Volterra numerical simulation method to determine the parameters of the model and its chaotic solution. This will allow the model to better reflect the actual situation as well as investigate new dynamic relationships with other economic variables.

References

[1] R. Nelson and S. Winter, An Evolutionary Theory of Economic change, Cambridge, Massachussetts and London, England: The Belknap Press of Harvard University Press Cambridge, 1982.

[2] E. S. Andersen, Evolutionary Economics : Post-Schumpeterian Contributions., i. e. edition, Ed., London: Thomson Learning, 1994, pp. 1-250.

[3] M. Yildizoglu, "Modeling Adaptive Learning : R&D Strategies in the model of Nelson & Winter (1982)," FREDE-E3i., Bordeaux, 2001.

[4] S. Redding, "The Low-Skill, Low-Quality Trap: Strategic Complementarities between Human Capital and R&D," Economic Journal, vol. 106, pp. 458-470, 1996.

[5] B. H. Tsai, C. S. Hsu and B. K. Balachandran, "Modeling Competition Between Mobile and Desktop Personal Computer LCD Panels Based on Segment Reporting Sales Information.," Journal of Accounting, Auditing & Finance, vol. 28, no. 3, pp. 273-291, 2013.

[6] B. H. Tsai and Y. Li, "Modelling competition in global LCD TV industry," Applied Economics, vol. 43, no. 22, pp. 2969-2981, 2011.

[7] R. Solow, "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics, vol. 70, no. 1, pp. 65-94, 1956.

[8] A. J. Lotka, Elements of physical biology, Baltimore, Williams & Wilkins Company., 1925.

[9] U. Dieckmann, P. Marrow and R. Law, "Evolutionary Cycling in Predator-Prey Interactions: Population Dynamics and the Red Queen," Journal of theoretical biology, vol. 176, no. 1, pp. 91-102, 1995.

[10] A. Castiaux, " Radical innovation in established organizations: Being a knowledge predator," Journal of Engineering and Technology management, vol. 24, no. 1-2, pp. 36-52, 2007.

[11] S. Morris and D. Pratt, "Analysis of the Lotka-Volterra competition equations as a technological substitution model," Technological Forecasting and Social Change, vol. 70, no. 2, pp. 103-133, 2003.

[12] T. Modis, "Technological Forecasting at the Stock Market," Technological Forecasting and Social Change, vol. 62, no. 3, pp. 173-202, 1999.

[13] S. J. Lee, D. J. Lee and H. S. Oh, "Technological forecasting at the Korean stock market: A dynamic competition analysis using Lotka-Volterra model," Technological Forecasting and Social, vol. 72, no. 8, pp. 1044-1057, 2005.

[14] S. M. Maurer and B. A. Huberman, "Competitive Dynamics of Web Sites," Journal of Economic Dynamics and Control, vol. 27, no. 11, pp. 2195-2206, 2003.

[15] J. Kim, D. J. Lee and J. Ahn, "A dynamic competition analysis on the Korean mobile phone market using competitive diffusion model," Computers & Industrial Engineering, vol. 51, no. 1, pp. 174-182, 2006.

[16] P. Parker and H. Gatignon, "Specifying competitive effects in diffusion model : empirical analysis," International Journal of Research in Marketing, vol. 11, no. 1, pp. 17-39, 1994.

[17] D. Delfino and P. J. Simmons, Positive and Normative Issues of Economic Growth, The University of York, 2000.

[18] A. R. Nazemi and S. Effati, "An application of a merit function for solving convex programming problems," Computers and Industrial Engineering, vol. 66, no. 2, pp. 212-221, 2013.

[19] E. P. Odum and G. W. Barrett, Fundamentals of ecology, Belmont CA: Thomson Brooks/Cole, 2005.

[20] S. Y. Chiang and G. G. Wong, "Competitive diffusion of personal computer shipments in Taiwan," Technological Forecasting and Social Change, vol. 78, no. 3, pp. 526-535, 2011.

[21] K. Smitalova and S. Sujan, A Mathematical Treatment of Dynamical Models in Biological Science, West Sussex UK: Eills Horwood, 1991.

[22] F. Wu, X. Mao and J. Yin, "Uncertainty and economic growth in a stochastic R&D model," Economic Modelling, vol. 25, no. 6, pp. 1306-1317, 2008.

[23] M. L. Tushman and P. Anderson, "Technological Discontinuities and Organizational Environments," Administrative Science Quarterly, vol. 31, no. 3, pp. 439-465, 1986.

[24] D. H. Boucher, S. James and K. H. Keeler, "The Ecology of Mutualism," Annual Review of Ecology and Systematics, vol. 13, no. 1, pp. 315-347, 1982.

[25] M. B. Jensen, B. Johnson, E. Lorenz and B. A. Lundvall, "Forms of Knowledge and Modes of Innovation," Research Policy, vol. 36, pp. 680-693, 2007.

[26] Y.H. Ko, "The Asymptotic Stability behaviour in a Lotka-Volterra Type Predator-Prey System," Bulletin of Korean Mathematical Society, vol. 43, pp. 575-587, 2006.

[27] P. M. Romer, "Endogenous technological change," Journal of Political Economy, vol. 98, no. 5, pp. S71-

S102, 1990.

[28] C. I. Jones, “R&D-based models of economic growth,” Journal of Political Economy , vol. 103, no. 4, pp. 759-784, 1995.

[29] V. Volterra, “Fluctuations in the Abundance of a Species considered Mathematically,” Nature, vol. 118, no. 2972, pp. 558-560, 1926.

[30] I. M. Bomze, “Lotka-Volterra equation and replicator dynamics: new issues in classification,” Biological Cybernetics, vol. 72, no. 5, pp. 447-453, 1995.

[31] O. Gurel and L. Lapidus, “Stability via Liapunov's Second Method,” Industrial Engineering Chemistry, vol. 60, no. 6, p. 12–26, 1968.

[32] B. S. Goh, “Global Stability in Many-Species Systems,” The American Naturalist, vol. 111, no. 977 , pp. 135-143, 1977.

[33] A. Korobeinikov, “A Lyapunov function for Leslie-Gower predator-prey models,” Applied Mathematics Letters, vol. 14, no. 6, pp. 697-699, 2001.

[34] M. W. HIRSCH and S. SMALE, Differential Equations, Dynamical Sykstems, and Linear Algebra, San Diego California 92101: ACADEMIC PRESS INC, 1974.

[35] A. Korobeinikov and W. T. Lee, “Global asymptotic properties for a Leslie-Gower food chain model,” Mathematical Biosciences and Engineering, vol. 6, no. 3, pp. 585-590, 2009.

[36] C. Vargas-De-León, “Lyapunov functions for two-species cooperative systems,” Applied Mathematics and Computation, vol. 219, no. 5, p. 2493–2497, 2012.

[37] T. Modis, “Genetic re-engineering of corporations,” Technological Forecasting and Social Change, vol. 56, no. 2, pp. 107-118, 1997.

[38] A. Lotka, Elements of Mathematical Biology, Dover Publications Inc. , 1956.

[39] B. Å. Lundvall and J. Björn, “The learning economy,” Journal of Industry Studies , vol. 1, no. 2, p. 23–42, 1994.

7. Appendix : Proof of the stability of S_δ

Considering that $S_\delta = \{x, y \in IR^2 \;\; x > 0, y > -\delta\}$

The boundary of S_δ is made up of two half-levels D_l and D_2 with,

$D_l = \{x, y \in IR^2 \;\; x > 0, y = -\delta\}$ and $D_2 = \{x, y \in IR^2 \;\; x = 0, y > -\delta\}$

Taking (x,y) a point in D_1, we thus have, $\vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $f(x,y)= \begin{pmatrix} x(\theta x + \xi\delta + \theta n) \\ 0 \end{pmatrix}$

It is therefore clear that $f(x, y) . \vec{n} = 0$.

On the other hand, if *(x,y)* is taken as a point in D_2 we have, $\vec{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $f(x,y) = \begin{pmatrix} 0 \\ (y+\delta)\alpha'(-y+n) \end{pmatrix}$

The same reasoning gives us $f(x, y).\ \vec{n} = 0$.

The proof of the stability of S_δ is based on the following theorem:

Theorem : Assuming S is a regular closed area without borders on IR²

$$\begin{cases} \dot{x}(t) = f(x,y,t) \\ \dot{y}(t) = g(x,y,t) \end{cases} \qquad (1)$$

Where *f* and *g* are continuously differentials. Supposing that $\vec{n}$ is a normal vector coming from the area S at the point *(x,y)*, and $\forall (x,y) \in S$ we have: $\vec{n}\,.<\dot{x},\dot{y}> = 0$. Therefore *S* is invariable [24] with respect to system *(1)*. This theorem is a classic result of differential systems, citing the title [17].

Hence the application of the above theorem allows us to say that S_δ is an invariant, that is to say if $(x(0), y(0)) \in S_\delta$.

Then $(x(t), y(t)) \in S_\delta$.

[24] In general, a surface S is invariant with respect to a system of differential equations if every solution that stats on S does not escape S.