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# About the Proof of the L'Hôpital's Rule

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**Abstract**

The recent proofs of L'Hopital's rule require the continuity of the derivatives of the functions in an interval. In this work, a proof is given so that it does not require the existence of the derivatives of the functions in any other number than the number where the limit of its ratio is calculated, more precisely the L'Hopital rule extends to locally defined and derivable functions in the number where the limit of the ratio of the functions is calculated.

***Keywords:*** Locally defined; L'Hopital's rule; Limit.

## 1. Introduction

The present L'Hopital Rule proof is based on the algebra of limits of locally defined functions in the number where the limit is calculated, as it is done for the proofs of the derivation rules where the functions involved need to be defined locally and derivable only in the number where the limit is calculated. Consequently, this proof extends the L'Hopital rule to locally defined and derivable functions only in the number where the limit of the ratio of the same is calculated.The ratio for the differences of the given functions will be considered, which will be expressed as a ratio of derivation quotients of each of the two functions, which by their definition tend to the derivatives of the functions, respectively.The derivation quotient of a given function is the one that is used to calculate the derivative of it. This quotient is locally defined and always different from zero when the derivative is different from zero in the number where the limit is calculated, so that the ratio of the differences of the functions (e.g. [4]), can be expressed as a ratio of derivation quotients.

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**2. Definitions y results on continuity and derivation of locally defined functions**

For our purposes it will be necessary to define what means locally defined functions in a given number and their derivation quotients, for which it is necessary to introduce some basic auxiliary concepts and results.

*Definition 1. Given a number* $a \neq 0$ *in te interval* $(-\infty, +\infty)$ , $sgn(a) = \frac{a}{|a|}$.

For our purposes we define te norm of a couple of numbers.

*Definition* 2. *Given a pair of numbers* $(a, b)$, *denote* $a^2 + b^2$ *by* $\|a, b\|^2 = a^2 + b^2$.

*Remark* 3. $sgn(b) - sgn(a) = \begin{cases} 0, si\ sgn(b) = sgn(a) \\ \pm 2, si\ sgn(b) \neq sgn(a) \end{cases}$ *and for any pair of numbers* $a, b$ *:* $|a|, |b| \leq \|a, b\|$.

*Proposition 4. Assume that* $|a - b| < \varepsilon < |a|$ *then* $sgn(a) = sgn(b)$

*Proof.* $\mathrm{sgn}(a) - \mathrm{sgn}(b) = \frac{a}{|a|} - \frac{b}{|b|} = \frac{|b|a - |a|b}{|a||b|} = \frac{|b|(a-b) + (|b| - |a|)b}{|a||b|}$ then

$|\mathrm{sgn}(a) - \mathrm{sgn}(b)| \leq \frac{|b||a-b| + ||b| - |a||b}{|a||b|} \leq \frac{|b|}{|a||b|}|a - b| \leq \frac{|a-b|}{|a|} < \frac{|a-b|}{\varepsilon} < 1$. Because of the observation

3, we have $\mathrm{sgn}(a) = \mathrm{sgn}(b)$.

*Proposition 5. Assuming that* $\varepsilon < |a|$ , $|b - a| < \varepsilon$ *sí y sí y solo si* $0 < |a| - \varepsilon < b < |a| + \varepsilon$.

*Proof.* : $0 < \varepsilon^2 - (b - a)^2 = \varepsilon^2 - b^2 - a^2 + 2ab$, that is to say $\|a, b\|^2 - \varepsilon^2 < 2ab$ then by the assumption about the norm and the observation 3, $ab > 0$, if and only if $ab = |ab| = |a||b|$ if and only if $0 < \varepsilon^2 - b^2 - a^2 + 2ab = \varepsilon^2 - |b|^2 - |a|^2 + 2|a||b| = \varepsilon^2 - (|b| - |a|)^2$ if and only if $\left||b| - |a|\right| < \varepsilon$ if and only if $-\varepsilon + |a| < b < \varepsilon + |a|$ . Reciprocally, if $0 < |a| - \varepsilon < b < |a| + \varepsilon$ by te supposition and the preceding proposition $\mathrm{sgn}(a) = \mathrm{sgn}(b)$ if and only if $ab = |ab|$ if and only if $|a - b| < \varepsilon$.

We will consider functions defined in te subsets of the interval $(-\infty, +\infty)$ and with the range in the same interval.

*Definition* 6. *A number in the domain of a function is said to be a pre-image or simply a number of the given function.*

*Definition* 7. *A number in the range of a function is called a value of the given function.*

*Definition* 8. *It is said that* $x \neq a$ *is close to* $a$ *if and only if* $|x - a| < \delta$ *for some* $\delta = \delta(a)$ .

The functions considered here do not need to be defined globally.

*Definition* 9. *A function is said to be locally defined in a given number if it is defined for all pre-images (numbers of the function) close to the number $a$. Example 10. The function $x \mapsto sgn(x)$ is locally defined at 0, since it is defined for $x \neq 0$ and close to* 0. *Definition* 11. *Let $g$ be defined locally in a number $a$. It is said that the limit of such function exists in the given number $a$ when there is a number $l$ such that for any positive number $\varepsilon$ there exists $\delta$ such that $|g(x) - l| < \varepsilon$ provided that $|x - a| < \delta$: the values of $g$ are arbitrarily close to $l$ for all the numbers of the function sufficiently close to $a$. Remark 12. If $\lim_{x\to a} f(x)$ exists, then $\tilde{f}(x) = \begin{cases} f(x), x \neq a \\ \lim_{x\to a} f(x), x = a \end{cases}$. is continuous at $a$ in which case it is asserted that $f$ is continuously extended to $a$.*

*Definition* 13.*The derivation quotient of a function $f$ locally defined at the number $a$ is the ratio of differences $\frac{f(x)-f(a)}{x-a}$.*

*Proposition 14. There exists $g'(a)$ and it is not null if and only if there exists $\left(\frac{x-a}{g(x)-g(a)}\right)(a)$ and it is null.*

*Proof.* By the definition of derivative: $g'(a) = \lim_{x\mapsto a} \frac{g(x)-g(a)}{x-a}$ and by the definition of limit $x \mapsto \frac{g(x)-g(a)}{x-a}$ it is locally defined at $a$ and $\left|\frac{g(x)-g(a)}{x-a} - g'(a)\right| < \varepsilon$ for any $x$ sufficiently close to $a$. By Proposition 6 we have that if $\varepsilon < |g'(a)|$, then $0 < |g'(a)| - \varepsilon < \frac{g(x)-g(a)}{x-a}$ for $x$ sufficiently close to $a$. Consequently, the quotient $\frac{x-a}{g(x)-g(a)}$ is defined for esta $x$ sufficiently close to $a$. By the definition of derivative: $\left(\frac{x-a}{g(x)-g(a)}\right) =$

$$\lim_{x\to a}\left(\frac{\frac{x-a}{g(x)-g(a)} - \lim_{x\to a}\frac{x-a}{g(x)-g(a)}}{x-a}\right) = \lim_{x\to a}\left(\frac{\frac{x-a}{g(x)-g(a)} - \lim_{x\to a}\frac{1}{\frac{g(x)-g(a)}{x-a}}}{x-a}\right)$$

$$= \lim_{x\to a}\left(\frac{\frac{x-a}{g(x)-g(a)} - \frac{1}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}}{x-a}\right) = \lim_{x\to a}\left(\frac{\frac{x-a}{g(x)-g(a)} - \frac{1}{g'(a)}}{x-a}\right) = \lim_{x\to a}\left(\frac{\frac{(x-a)g'(a) - \big(g(x)-g(a)\big)}{\big(g(x)-g(a)\big)g'(a)}}{x-a}\right)$$

$$= \lim_{x\to a}\left(\frac{\frac{(x-a)g'(a) - \big(g(x)-g(a)\big)}{\big(g(x)-g(a)\big)g'(a)}}{x-a}\right) = \lim_{x\to a}\left((x-a)\frac{g'(a) - \frac{g(x)-g(a)}{x-a}}{\frac{g(x)-g(a)}{(x-a)}g'(a)}\right) = \lim_{x\to a}\left((x-a)\frac{g'(a) - \frac{g(x)-g(a)}{x-a}}{\frac{g(x)-g(a)}{(x-a)}g'(a)}\right)$$

$$= \lim_{x\to a}(x-a)\lim_{x\to a}\frac{g'(a) - \frac{g(x)-g(a)}{x-a}}{\frac{g(x)-g(a)}{(x-a)}g'(a)} = 0\,\frac{g'(a) - \lim_{x\to a}\frac{g(x)-g(a)}{x-a}}{\left(\lim_{x\to a}\frac{g(x)-g(a)}{(x-a)}\right)g'(a)} = 0\,\frac{g'(a)-g'(a)}{g'(a)g'(a)} = 0(0) = 0$$

Reciprocally, by te definition of $\left(\frac{x-a}{g(x)-g(a)}\right)(a)$ and the definition of limit $x \to \frac{\frac{x-a}{g(x)-g(a)} - \frac{1}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}}{x-a}$ it is definied and $\left|\frac{\frac{x-a}{g(x)-g(a)} - \lim_{x\to a}\frac{x-a}{g(x)-g(a)-}}{x-a}\right| < \varepsilon$ for $x$ sufficiently close to $a$ if and only if there exists$\lim_{x\to a}\frac{1}{\frac{g(x)-g(a)}{x-a}} = \frac{1}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}$ if and only if $g'(a)$ there exists and its value is not null.

**3. Main Results**

The derivation quotients are the main ingredient for the demonstration of the L'Hopital Rule presented here.

*Theorem 15. If $g'(a)$ exists and it is not null, then $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$ exists if and only if $f'(a)$ exists. In this case $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'}{g'}(a)$.*

*Proof.* By the definition of derivative and limit, Proposition 13 implies that $x \to \frac{f(x)-f(a)}{x-a} = \frac{g(x)-g(a)}{x-a}\big(f(x) - f(a)\big)\frac{1}{g(x)-g(a)} = \frac{f(x)-f(a)}{x-a}\frac{g(x)-g(a)}{g(x)-g(a)}$ it is defined for $x$ close to $a$ as the product of the functions $x \to \frac{g(x)-g(a)}{x-a}, \frac{f(x)-f(a)}{g(x)-g(a)}$, since by hypotesis there is a derivative of $g$ and the limit of the quotient of the differences in $a$ of the functions $f$ y $g$ in $a$. By the Algebra of limits $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{\substack{x\to a\\ x\to a}}\frac{g(x)-g(a)}{x-a}\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$, that is, there is a derivative of $f$ in $a$ and consequently $f'(a) = g'(a)\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$ and in this case $\frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$. Recíprocally, the functions $x \to \frac{g(x)-g(a)}{x-a}, \frac{f(x)-f(a)}{x-a}$ are defined for $x$ sufficiently close to $a$ by Proposition 13 and the definition of derivative and in this case $\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)-f(a)}{x-a}\frac{x-a}{g(x)-g(a)} = \frac{f(x)-f(a)}{x-a}\frac{1}{\frac{g(x)-g(a)}{x-a}} = \frac{f(x)-f(a)}{x-a}\frac{1}{\frac{g(x)-g(a)}{x-a}} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$ , then by the algebra of limits, we have $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \frac{f'}{g'}(a)$.

*Corollary 16. If $f(a) = g(a) = 0$ and also $g'(a)$ exists and is not null, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists if and only if $f'(a)$ exists. In this case $\lim_{x\to a}\frac{f}{g}(x) = \frac{f'}{g'}(a)$*

*Proof.* Replacing the present hypothesis, that is, $f(a) = g(a) = 0$, in Theorem 15, we have the result of the present Corollary.

**4. Discussion**

Several authors such as [1; 25] require the assumption of the continuity of the derivatives while in this work, such assumption is not necessary, causing the spectrum of application of this rule to be extended.

## 5. Conclusions

It is well known that the L'Hopital rule proof depends directly on the Roll Theorem or the mean value theorem, that is, it is necessary to assume the global continuity of the derivative (even where the logic of the proof of the considered rule (see for example.[2]). It has been established, the rule of L'Höpital as it is established the rules of derivation, that is to say the rule of H'ôpital can be considered that it becomes one more rule of derivation. The substitution of the ratio of the derivatives by the limit of the ratio of the derivatives to the number where the limit will be calculated, actually requires that the derivative of the functions is defined in an interval contained in the domain of the functions besides require that it be extended continuously to the number where the limit is calculated. In general, the condition of local continuity has been replaced by a sophisticated global condition in intervals of the number line, such as the existence of the derivative in some interval [2,18] increasing ratios of functions [1,3,16,23], quotients of functions of C1 [3,15], ratios of series of numbers [8,19,25], ratios of functions of Darboux and / or generalized derivation [9,21,24], functions absolutely continuous and / or Frechet differentiable [10,12], analytic function quotients [11,17,20], quotients of integrable functions [14,20,22]. This demonstration can be used with baccalaureate students as a rule to calculate limits such as $\lim_{x\to 0}\frac{\exp\left(-\frac{1}{x}\right)}{\sin x} = \frac{\left(\exp-\frac{1}{x}\right)(0)}{\sin\prime 0} = \frac{0}{\cos 0} = 0$, in terms of the calculation of derivatives. where the derivative $\left(\exp-\frac{1}{x}\right)(0)$ has been calculated with the definition of derivative.

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