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Approximation Theory on Summability of Fourier Series

Sanjay Mukherjee[a]*, A J Khan[b]

[a]*Research Scholor, MATS University, Raipur(C.G.), India*

[b]*MATS University, Raipur(C.G.), India*

[a]*Email: sanjaymukherjeeruma@gmail.com*

[b]*Email: khanaj@matsuniversity.ac.in*

Abstract

The results of Chandra to (e,c) means U.K.Shrivastava and S.K.Verma have proved the following theorem

THEOREM : Let $f \in C_{2\pi} \cap Lip \propto, 0 < \propto \leq 1$. Then

$\|t_n^c - f\| = o\left(n^{-\propto/2}\right)$,

Where $t_n^c(f;x)$ is nth (e, c) means of fourier series of f at x.

In this paper we obtain the Fourier series by (N,p,q)(E,1) which is the analogues to the (e , c) means given above .The theorem is as follows

THEOREM: Let $\{p_n\}$ and $\{q_n\}$ be the positive monotonic, non increasing sequence of real numbers be summable (N,p,q)(E,1) to f(x) at the point t=x is

$t_N^{p,q,E} - f(x) = o(1)$

Keywords: Fourier series; Borel means; Lebesgue series.

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* Corresponding author.

1. Introduction

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex, such that

$$P_n = p_1 + p_2 + p_3 + \cdots p_n = \sum_{r=0}^{n} p_r \to \infty, as\ n \to \infty,$$

$$Q_n = q_1 + q_2 + q_3 + \cdots q_n = \sum_{r=0}^{n} q_r \to \infty, as\ n \to \infty, \tag{1.1}$$

$$R_n = p_0q_n + p_1q_{n-1} + p_3q_{n-2} + \cdots p_nq_0 = \sum_{r=0}^{n} p_rq_{n-r} \to \infty, as\ n \to \infty$$

Given two sequences $\{p_n\}$ and $\{q_n\}$ convolution $(p * q)$ is defined as

$$R_n = (p \neq q)_n = \sum_{r=0}^{n} p_{n-r}\, q_r \tag{1.2}$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its nth partial sums$\{s_n\}$.

We write $t_n^{p,q} = \frac{1}{R_n}\sum_{r=0}^{n} p_{n-r}q_r$ (1.3)

If $R_n \neq 0$, for all n, the generalized Norlund transform of the sequence $\{s_n\}$ is the sequence$\{t_n^{p.q}\}$.

If $t_n^{p,q} \to S, as\ n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is summable to S by

$$S_n \to S(N, p, q) \tag{1.4}$$

The necessary and sufficient conditions for (N,p,q) method to be regular are

$$\sum_{r=0}^{n}|p_{n-r}q_r| = o(|R_n|) \tag{1.5}$$

And $p_{n-r} = o(|R_n|)$, as $n \to \infty$ for every fixed $k \geq 0$, for which $q_r \neq 0$

$$E_n^1 = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r} s_r \tag{1.6}$$

If $E_n^1 \to s, as\ n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be (E,1) summable to s (Hardy [1]) :

$$t_n^{p,q,E} = \frac{1}{R_n}\sum_{r-0}^{n} p_{n-r}q_rE_r^1$$

$$= \frac{1}{R_n}\sum_{r=0}^{n} p_{n-r}q_r \frac{1}{2^k}\sum_{r=0}^{n}\binom{k}{r}s_r \tag{1.7}$$

If $T_n^{p,q,E} \to \infty, as\ n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is summable to S by

(N,p,q)(E,1) summability method.

2. Structure

2. Degree of approximation by borel means and (E, Q) means were obtained by Chandra [4] and [5] respectively .Extending the results of Chandra to (e,c) means U.K.Shrivastava and S.K.Verma[9] have proved the following theorem

THEOREM : Let $f \in C_{2\pi} \cap Lip \propto, 0 < \propto \leq 1$. Then

$\|t_n^c - f\| = o\left(n^{-\propto/2}\right),$

Where $t_n^c(f; x)$ is nth (e,c) means of fourier series of f at x. (2.1)

Our theorem fourier series by (N,p,q)(E,1) is the analogues to the (e,c) means theorem, which is as follows

THEOREM: Let $\{p_n\}$ and $\{q_n\}$ be the positive monotonic ,non increasing sequence of real numbers be summable (N,p,q)(E,1) to f(x) at the point t=x is

$t_N^{p,q,E} - f(x) = o(1)$

Proof of the above theorem required some lemmas

3. Lemmas

Lemma 3.1- For $0 \leq t \leq \frac{1}{n}$ $|K_n(t)| = o(n)$

Lemma 3.2- If $\{p_n\}$ and $\{q_n\}$ are non negative and non increasing, then for $0 \leq a \leq b < \infty, 0 \leq t \leq \pi,$ and any n we have $\frac{1}{2\pi R_n}\left|\sum_{r=a}^{b} p_{n-r} q_r \frac{cos^r(t/2) sin(r+1)(t/2)}{sin(t/2)}\right| = o\left(\frac{R_k}{tR_n}\right)$

4. Proof of Theorem

Let f(t) be a periodic function with period 2π and integrable in the same sense of Lebesgue over the interval $(-\pi, \pi)$

Let its Fourier series be given by

$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt)$ (4.1)

Following Zygmund [3] , the nth sum $s_n(x)$ of the series at t=x is given by

$$s_n(x) = f(x) + \frac{1}{2\pi}\int_0^{\pi} \emptyset_x(t)\, \frac{sin(n+1)t}{sin(t/2)}\, dt \tag{4.2}$$

So the (E,1) mean of the series at t=x is given by

$$E_n^1(x) = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r} s_r(x)$$

$$= f(x) + \frac{1}{2^{n+1}\pi}\int_{r=0}^{\pi} \frac{\emptyset_x(t)}{sin(t/2)}\left\{\sum_{r=0}^{n}\binom{n}{r} sin\left(r+\frac{1}{2}\right)t\right\}dt$$

$$= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi} \frac{\emptyset_x(t)}{sin(t/2)} Im\left\{e^{it/2}(1+e^{it})^n\right\}dt$$

$$= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi} \frac{\emptyset_x(t)}{sin(t/2)} Im\left\{e^{it/2}(1+cost+isint)^n\right\}dt \tag{4.3}$$

$$= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi} \frac{\emptyset_x(t)}{sin(t/2)} Im\left\{e^{it/2}2^n cos^n\left(\frac{t}{2}\right)\left(cos\frac{t}{2}+isin\frac{t}{2}\right)^n\right\}dt$$

$$= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi} \frac{\emptyset_x(t)}{sin(t/2)} Im\left\{e^{it/2}2^n cos^n\left(\frac{t}{2}\right)\left(cos\frac{nt}{2}+isin\frac{nt}{2}\right)\right\}dt$$

$$= f(x) + \frac{1}{2\pi}\int_0^{\pi} \emptyset_x(t)\frac{cos^n(t/2)sin(n+1)(t/2)}{\sin(\frac{t}{2})}dt$$

Therefore

$$t_n^{p,q,E}(x) - f(x) = \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right] K_n(t)\emptyset_x(t)dt$$

$$= I_1 + I_2 + I_3 \text{ (say)} \tag{4.4}$$

We have

$$|I_1| \le \int_0^{1/n} |K_n(t)|\,|\emptyset_x(t)|dt$$

$$= O(n)\int_0^{1/n}|\emptyset_x(t)|\, dt \ (\, using\ Lemma\ 3.1) \tag{4.5}$$

$$= o\left(\frac{1}{\alpha(n)}\right)$$

$$= o(1)\ as\ n \to \infty \tag{4.6}$$

Now

$$|I_2| \leq \int_{1/n}^{\delta} |K_n(t)|\,|\emptyset_x(x)| dt\ (where\ 0 < \delta < 1)$$

$$= \int_{1/n}^{\delta} o\left(\frac{R(1/t)}{tR(n)}\right) |\emptyset_x(t)|\, dt\ (using\ Lemma\ 3.2)$$

$$= o\left(\frac{1}{R(n)}\right) \int_{1/n}^{\delta} \left(\frac{R(1/t)}{t}\right) |\emptyset_x(t)|\, dt$$

$$= o\left(\frac{1}{R(n)}\right)\left[\left\{\frac{R(1/t)}{t}\emptyset_x(t)\right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} d\left(\frac{R(1/t)}{t}\right)\emptyset_x(t)\right]$$

$$= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right)\left[\int_{1/n}^{\delta} \emptyset_x(t)\left\{d\left(\frac{R(1/t)\alpha(1/t)}{t\alpha(1/t)}\right)\right\}\right]$$

$$= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1)$$

$$= o(1), as\ n \to \infty \tag{4.7}$$

Now

$$I_3 = \int_{\delta}^{\pi} |K_n(t)||\emptyset_x(t)|\, dt$$

By Riemann-Lebesgue theorem and regularity of the method of summability we have

$$I_3 = o(1), as\ n \to \infty \tag{4.8}$$

Combining (4.6),(4.7) and (4.8) we get

$t_N^{p,q,E} - f(x) = o(1)$

This completes the proof of the theorem.

5. Conclusion

We conclude that the above theorem which is proved in (e,c) means can be proved by (N,p,q)(E,1) means.

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References

[1] G.H. Hardy, Divergent series, Oxford University Press, Oxford,UK,1st edition ,1949.

[2] D. Borwein , On product of sequences ,Journal of the London Mathematical Society, vol.33pp.352-357, (1958).

[3] A. Zygmund , Trigonometric Series. Vol. I Cambridge University Press, Cambridge, UK, 2nd edition, 1959.

[4] P. Chandra. On the degree of approximation of function belonging to Lipschitz Class. Labdev. Jour. Science and Technology, 13A (1975), 181-183.

[5] P. Chandra. On the degree of approximation of continuous functions Commun. Fac. Sci. Univ. Ankara Ser A, 30(1981),7-16.

[6] K.Ikeno, Lebesgue constant for a family of Summability method, Tohoku Math. J.,17(1965),250-265.

[7] B.Kuttner, C.T.Rjajagopal, and M.S.Rangachari, Tauberian convergence theorems for summability methods of the Karamata family, J.Indian Math. Soc. 44(1980),23-38.

[8] A.Meir, Tauberian constants for a family of transformations, Annals of Math. 78(1963), 594-599

[9] U.K.Shrivastava and S.L.Varma, On the degree of approximation of function belonging to Lipschitz Class by (e,c) means, Tamkang Journal of Maths, 26,no.3(1965),97-101.