American Academic Scientific Research Journal for Engineering, Technology, and Sciences

ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

http://asrjetsjournal.org/

Construction of Alternative Axial Points Using Standard Axial Points of Central Composite Design

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Abstract

There has been an over-flogged attention given to propositions on how one can make good choice of the existing axial points rather than procedural techniques for constructing axial points about the existing axial points. In order to curb this oversight, this work has constructed axial points about the standard axial points. The construction has given rise to α^* = 0.99k (where k is the number of factors) in comparison to the standard axial points $\alpha = f^{\frac{1}{4}}$ (where f is the number of factorial points). Both axial points have been implemented on a central composite design used for maximizing a four-factor process. The constructed axial points produced yields of about 87.211%, better than the yield of 87.187% produced by the standard axial points. Furthermore, the central composite design resulting from the constructed axial points satisfied the D-, A- and E-optimality criteria in comparison to that obtained from the standard or existing axial points.

Keywords: Axial points; Optimality Criteria; Factorial points; Response Surface; Central composite design.

1. Introduction

Response Surface Methodology (RSM) is one of the frequently used statistical techniques for achieving process optimization. It was developed by [2]. RSM is a collection of mathematical and statistical techniques useful in modeling and analyzing a problem, where a set of controllable factors influence a response , the aim is to optimize the response,([11,12]) .Response Surface Methodology bases it methods on a supposed set of data containing observations, a response variable y and the independent variables ([10;9]). Response surface designs are designs used to model response surface. These designs can be classified as first-order or second-order design ([3;1]).

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First-order design is used to model a response surface when the current operating condition of the process is remote from its optimum [5]. Several first-order designs such as 2^k factorial, Plackett-Burman and simplex designs have been developed overtime; nevertheless, the most commonly used first-order design is the 2^k factorial [7]. In the course of implementing response surface methodology, the process engineer is eventually guided to the optimal region of his process via the method of steepest ascent (or descent) and least squares. In such circumstance, the need for a second-order model becomes imperative to account for system curvature; but fitting a second-order model requires the use of second-order design. Overtime, a variety of second re–order designs such as 3^k factorial design , central composite designs and Box-Behnken designs have been developed; the most commonly used remains the central composite design. Central composite design consists of 2^k factorial points, 2k axial points and n_0 center points. Overtime, many researchers such as [6,13], studied the central composite design in response surface analysis. These researchers opined that the selection of axial points is dependent on how many factorial points there are in the design; precisely, $\alpha = f^{\frac{1}{2}}$ is known to produce central composite designs that are effective with f denoting factorial points. In this research, we shall construct an alternative axial points about the standard axial points formulated by [2].

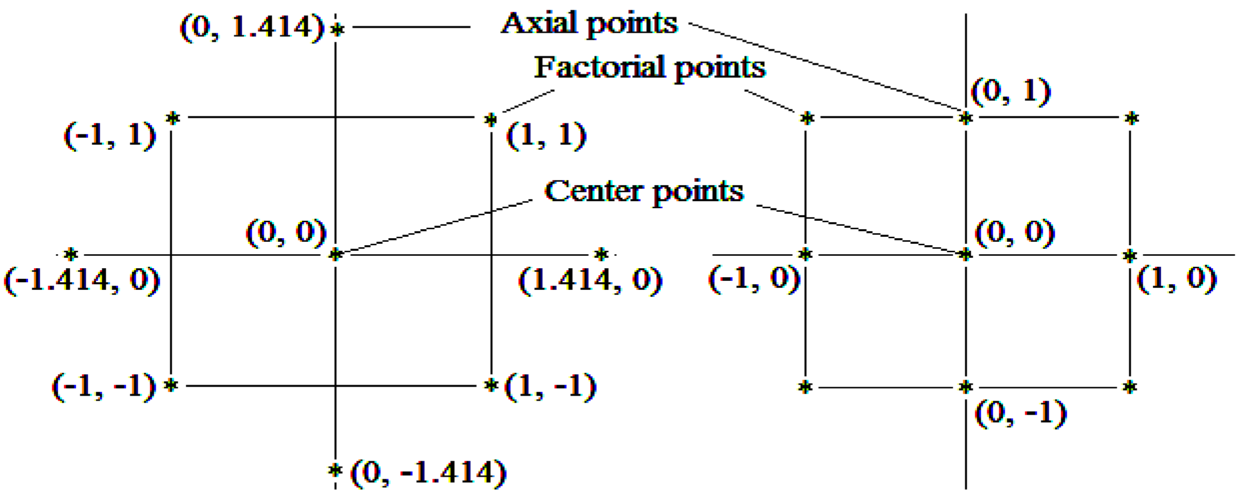
2. Material and Methods

2.1 Standard Central Composite Design

A Box-Wilson central composite design, commonly called central composite design contains an imbedded factorial or fractional design with center points that is augmented with a group of axial points that allow estimation of curvature. If the distance from the center of the design space to a factorial point is $\pm$ unit for each factor, the distance from the center of the design space to a star point is $|\alpha|>1$. The precise value of α depends on certain properties desired for the design and on the number of factors involved. Furthermore, the axial points are at some distance α from the center based on the properties desired for the design and the number of factors in the design. The axial points establish new extremes for the low and high settings for all factors. These designs have circular, spherical or hyper-spherical symmetry and also require 5 levels for each factor. Augment an existing factorial or resolution (v) fractional design with axial points can produce this design. Figure 1 shows a diagram of the standard central composite design, for those situations in which the limits specified for factor settings are truly limits, the inscribed centered central composite design uses the factor settings as the axial points and creates a factorial or fractional design within those limits; in other words, an inscribed central composite design is a scaled down standard central composite design with the level of central composite design divided by α. This design also requires 5 levels of each factor. **Figure 2** shows an inscribed central composite design.

2.2 Face-centered central composite design

In the face-centered central composite design, the axial points are at the center of each face of the factorial space, so that $\alpha = \pm 1$. This variant of the standard central composite design requires 3 levels of each factor. However, augmenting an existing factorial or a resolution fractional v factorial design with appropriate axial points can also produce this design. **Figure 3** depicts the face-centered central composite design.



Figures 1, 2: Standard central composite design (circumscribed central composite design) and face-centred cube.

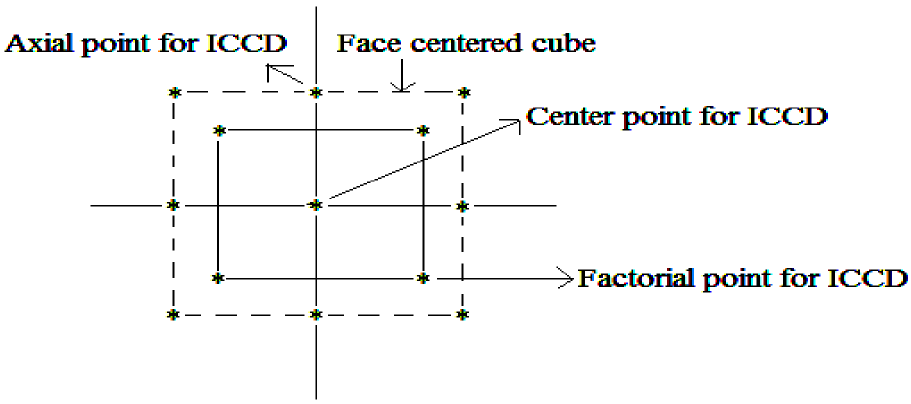


Figure 3: Inscribed central composite design (ICCD).

2.2 Rotatable design

An experimental design is said to be rotatable if the variance of the predicted response $\hat{y}$ is equal at all points equidistance from the design center. A design categorizing with the attribute leaving $v(\hat{y})$ constant shows that the design is rotatable about the center. Central composite design is rotatable if $\alpha = f^{\frac{1}{4}}$ with λ_α times observed by individual star point. To this, the design is rotatable if $\alpha = \left(\frac{f}{\lambda_\alpha}\right)^{\frac{1}{4}}$

2.3 Optimality Criteria for Testing Design Efficiency

An optimality Criterion is a single number that summarizes how good a design is, and it is maximized or minimized by an optimal design. The D-optimality criterion maximizes the determinant of the Information matrix X^TX or minimizes the determinant of the dispersion matrix $(X^TX)^{-1}$. Symbolically, a design is D-optimal if it gives $\max\{\det(X^TX)\}$ or $\min\{\det(X^TX)^{-1}\}$. The G-optimality criterion minimizes the maximum variance of the estimated response function min{max d(X, ζ)} over the design region. The experimenter optimizing a design according to the G-optimality criterion intends to get a good estimate of all the observed responses. The A-optimality criterion minimizes the trace of the dispersion matrix $(X^T X)^{-1}$. Symbolically, A-optimal design is a design that gives min{trace$(X^T X)^{-1}$}. The E-optimality criterion minimizes the maximum eigenvalue of the dispersion matrix $(X^T X)^{-1}$. Symbolically, A-optimal design is a design that gives min{trace($(X^T X)^{-1}$}. The E-optimality criterion minimizes the maximum eigenvalue of the dispersion matrix $(X^T X)^{-1}$. Symbolically, a design is E-optimal if it gives minλ^{-1}, where $\boldsymbol{\lambda}$ is the largest eigenvalue of the information matrix $X^T X$.

2.4 Procedure for construction and testing axial points

The following steps are used for the construction and testing of axial points:

(i) Draw lines from the center points of the design to each factorial point.
(ii) Construct perpendiculars to straight lines drawn to the factorial points from the center point.
(iii) Mark out the points of intersection on the star or axial axes.
(iv) join the constructed axial points to form a crystal (see figure 4)
(v) The construction gives rise to $\alpha^* =$ 1.988 for a two-factor case-study. By proportionality, a one-factor case study gives $\alpha^* = \frac{\alpha^*}{2} = \frac{1.988}{2}$ = 0.99

But for a three-factor case study $\alpha^* =$ o.99 x3 = 2.97. As a generalization, we therefore have $\alpha^* =$ 0.99k, for k factors.

(vi) Run central composite design on research case study using the standard axial points and the constructed axial points.
(vii) Compare results and established properties.

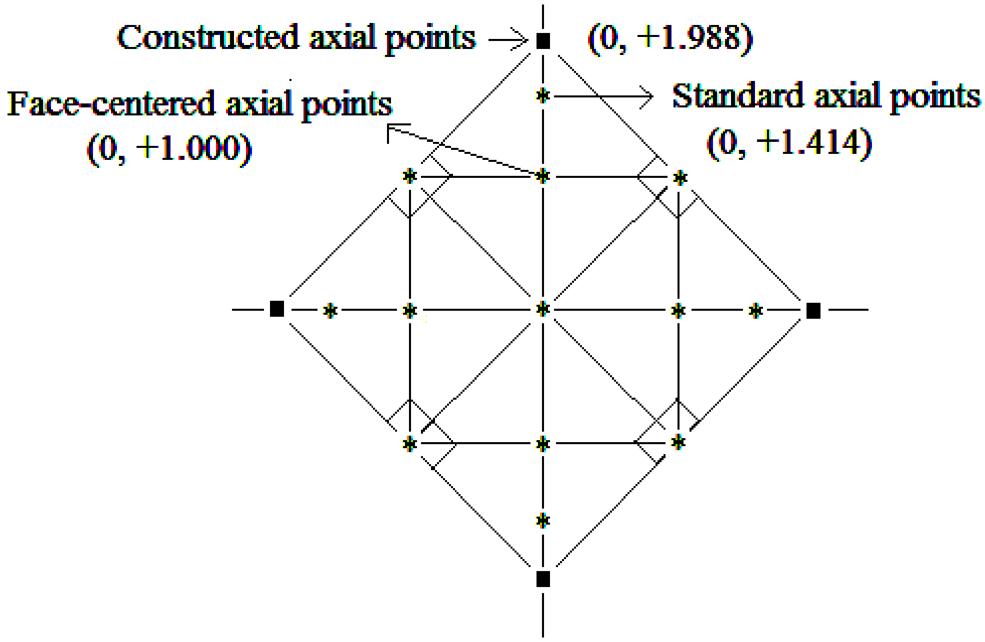


Figure 4: Diagram showing the constructed axial points.

3. Results and Discussion

3.1 Numerical illustration

Case study (Success Foods International, Calabar Municipal Council)

The data used was from food success International (Calabar, Municipal Council). The chemical engineer was interested in determining the operating conditions that improves the yield of his process. Four controllable factors influenced process yield. The factors are: temperature, pressure, concentration, and stirring rate. A factorial experiment was carried out in the pilot plant to study how these factors influenced the percentage yield of the product. He was operating the process at an operating condition around a reaction temperature of 24^0 Fahrenheit, reaction pressure of 35 atmosphere, percentage concentration of 155 percent and stirring rate of 75 percent which resulted in yields around 72 percent. Since it was unlikely that this region contained the optimum, a first order model was fitted and the method of steepest ascent applied. He decided that the region of exploration for fitting the first order model should be (19, 29) degrees Fahrenheit, (30, 40) atmosphere of pressure, (150, 160) percent of concentration and (70, 80) percent of stirring rate.

To simplify the calculations, we coded the independent variables to a (-1, 1) interval. Thus, if ξ_1 denotes the natural variable temperature, ξ_2 denotes the natural variable pressure, ξ_3 denotes the natural variable concentration and ξ_4 denotes the natural variable stirring rate then the coded variables are:

$$x_1 = \frac{\xi_1 - 24}{5} \quad x_2 = \frac{\xi_2 - 35}{5}, \quad x_3 = \frac{\xi_3 - 155}{5}, \quad x_4 = \frac{\xi_4 - 75}{5}.$$

The data is displayed in **Table1 below**

Table 1: Process data for fitting the first first-order model.

Natural variables				Coded variables				Responses
ξ_1	ξ_2	ξ_3	ξ_4	x_1	x_2	x_3	x_4	y
19	30	150	70	-1	-1	-1	-1	72.3
19	30	150	80	-1	-1	-1	1	72.8
19	30	160	80	-1	-1	1	1	71.2
19	40	160	80	-1	1	1	1	72.9
29	40	160	80	1	1	1	1	73.1
29	40	160	70	1	1	1	-1	71.9
29	40	150	70	1	1	-1	-1	70.6
29	30	150	70	1	-1	-1	-1	69.9
29	30	160	70	1	-1	1	-1	70.9
19	40	150	80	-1	1	-1	1	67.9
29	30	150	80	1	-1	-1	1	69.9
19	40	160	70	-1	1	1	-1	71.9
29	40	150	80	1	1	-1	1	72.9
29	30	160	80	1	-1	1	1	73.9
19	30	150	70	-1	-1	1	-1	72.9
19	40	150	70	-1	1	-1	-1	68.9
24	35	155	75	0	0	0	0	72.9
24	35	155	75	0	0	0	0	72.9
24	35	155	75	0	0	0	0	72.6
24	35	155	75	0	0	0	0	72.7
24	35	155	75	0	0	0	0	72.8
24	35	155	75	0	0	0	0	72.7
24	35	155	75	0	0	0	0	72.8
24	35	155	75	0	0	0	0	72.9
24	35	155	75	0	0	0	0	72.8
24	35	155	75	0	0	0	0	72.6
24	35	155	75	0	0	0	0	72.8
24	35	155	75	0	0	0	0	72.9

The design used to collect the data is a 2^k factorial augmented by 12 center points. Repeat observations at the center were used to estimate the experimental error. The design is centered about the current operating conditions for the process. Using **MINITAB**, a first order model was fitted to this data by least squares as displayed below. Since overall lack of fit indicates model adequacy of the first-order model. We continue the procedure along the path of steepest ascent. In order to shift from the center of the design on the path of steepest ascent a movement of 0.176 units is made in the direction of X_1, for all -0.199 unit in the direction of X_2, 0.812 unit in the direction of X_3 and 0.363 unit in the direction of X_4. Hence, the path of steepest passed the center of the design with slope $\frac{X_2}{X_1} = \frac{-0.199}{0.176}$ $\frac{X_3}{X_1} = \frac{0.812}{0.176}$, $\frac{X_4}{X_1} = \frac{0.363}{0.176}$

A basic step size of five minutes of temperature was employed. With knowledge of the relationship between ξ_1 and X_1 it was observed that a reaction time of five degrees Fahrenheit is the same as one step in the coded variable. This implies that the steps along the path of steepest ascent are: $\Delta X_1 = 1.0000$, $\Delta X_2 = \left(\frac{-0.199}{0.176}\right)\Delta X_1 = -1.1307$, $\Delta X_3 = \left(\frac{0.812}{0.176}\right)\Delta X_1 = 4.6136$, $\Delta X_4 = \left(\frac{0.363}{0.176}\right)\Delta X_1 = 2.0625$.

We computed points on the resulting path steepest ascent, observing responses per point until no obvious response increase was observed. **Table 2 below** shows our observations for response increased up to step five. But step six produced a response decrease.

Table 2: Steepest ascent procedure using the first first-order model.

	Coded Variables				Natural Variables				Response
	X_1	**X_2**	**X_3**	**X_4**	ξ_1	ξ_2	ξ_3	ξ_4	**Y**
Origin	0.000	0.000	0.000	0.000	24	35	155	75	72.0
Δ	1.000	-1.131	4.164	2.063					
OriginΔ	1.000	-1.131	4.164	2.063	29	29.347	175.818	83.313	73.5
Origin+2Δ	2.000	-2.261	8.327	4.125	34	23.693	196.636	95.625	78.5
Origin+3Δ	3.000	-3.392	12.491	6.188	39	18.040	217.454	105.938	79.6
Origin+4Δ	4.000	-4.523	16.654	8.250	44	12.386	238.272	116.250	81.2
Origin+5Δ	5.000	-5.654	20.818	10.313	49	6.733	259.090	126.563	87.3
Origin+6Δ	6.000	-6.784	24.982	12.375	54	1.079	279.908	136.875	79.4

Therefore, another first-order model had to be fitted within the region about the point

($\xi_1 = 49$, $\xi_2 = 6.7325$, $\xi_3 = 259.090$, $\xi_1 = 126.5625$). Exploration region about ξ_1 was (44, 54), (2,12) about ξ_2, (254, 264) about ξ_3, and (122, 132) about ξ_4. Thus the coded variables were:

$X_1 = \frac{\xi_1 - 49}{5}$, $X_2 = \frac{\xi_2 - 7}{5}$, $X_3 = \frac{\xi_3 - 259}{5}$, $X_4 = \frac{\xi_4 - 127}{5}$,

The same design and augmentation was employed (**see table 3 below**) . In collecting our data we employed a

2^4 factorial design technique using 12 center points to augment. In order to estimate the error of experiment, observations were repeated at the design center.

Table 3: Process data for fitting the second first-order model.

Natural variables				Coded variables				Responses
ξ_1	ξ_2	ξ_3	ξ_4	x_1	x_2	x_3	x_4	y
44	2	254	122	-1	-1	-1	-1	87.6
44	2	254	132	-1	-1	-1	1	87.5
44	2	264	132	-1	-1	1	1	87.4
44	12	264	132	-1	1	1	1	87.0
54	12	264	132	1	1	1	1	87.0
54	12	264	122	1	1	1	-1	86.6
54	12	254	122	1	1	-1	-1	87.1
54	2	254	122	1	-1	-1	-1	87.9
54	2	264	122	1	-1	1	-1	87.8
44	12	254	132	-1	1	-1	1	87.0
54	2	254	132	1	-1	-1	1	87.1
44	12	264	122	-1	1	1	-1	87.0
54	12	254	132	1	1	-1	1	87.0
54	2	264	132	1	-1	1	1	87.7
44	2	264	122	-1	-1	1	-1	87.5
44	12	254	122	-1	1	-1	-1	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.1
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.4
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.1
49	7	259	127	0	0	0	0	87.2

The lack of fit test (**see Appendix B**) indicates that the model does not fit the data at an overall level of significance of P =0.01. This curvature in the true surface indicates that we are near the optimum. At this point, additional analysis had to be done to locate the optimum more precisely. He cannot fit a second-order model in X_1, X_2 , X_3 and X_4 variables with the data in **Table 3.** So we decided to augment this data with more points to fit a second-order model. To get this done, we got four observation :

($X_1 \pm$ 2.000, $X_2 =$ 0, $X_3 =$ 0, $X_4 =$ 0); ($X_1 =$ 0, $X_2 \pm$ 2.000, $X_3 =$ 0, $X_4 =$ 0); ($X_1 =$ 0, $X_2 =$ 0, $X_3 \pm$ 2.000, $X_4 =$ 0)

($X_1 =$ 0, $X_2 =$ 0, X_3 = 0, X_4 $\pm$ 2.000). The complete data set is displayed in **Table 4 below.**

Table 4: Process data for fitting the second-order model using the standard axial point.

Natural variables				Coded variables				Responses
ξ_1	ξ_2	ξ_3	ξ_4	x_1	x_2	x_3	x_4	y
44	2	254	122	-1	-1	-1	-1	87.6
44	2	254	132	-1	-1	-1	1	87.5
44	2	264	132	-1	-1	1	1	87.4
44	12	264	132	-1	1	1	1	87.0
54	12	264	132	1	1	1	1	87.0
54	12	264	122	1	1	1	-1	86.6
54	12	254	122	1	1	-1	-1	87.1
54	2	254	122	1	-1	-1	-1	87.9
54	2	264	122	1	-1	1	-1	87.8
44	12	254	132	-1	1	-1	1	87.0
54	2	254	132	1	-1	-1	1	87.1
44	12	264	122	-1	1	1	-1	87.0
54	12	254	132	1	1	-1	1	87.0
54	2	264	132	1	-1	1	1	87.7
44	2	264	122	-1	-1	1	-1	87.5
44	12	254	122	-1	1	-1	-1	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.1
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.4
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.1
49	7	259	127	0	0	0	0	87.2
59	7	259	127	2.00	0	0	0	87.4
39	7	259	127	-2.00	0	0	0	87.1
49	17	259	127	0	2.00	0	0	87.4
49	-3	259	127	0	-2.00	0	0	87.3
49	7	269	127	0	0	2.00	0	87.5
49	7	249	127	0	0	-2.00	0	87.4
49	7	259	137	0	0	0	2.00	87.4
49	7	259	117	0	0	0	-2.00	87.3

The **Appendix C** shows the analysis done using **Table 4**, and which clearly indicates model adequacy of the second order model. In **Table 5 (APPENDIX A)** a display of the coded variables accompanied by the constructed axial points is presented. **Appendix D** shows the analysis done based on **Table 5**. Clearly the second-order model developed via **Table 5** is also adequate for explaining the curvature in the system. The construction produced an approximate axial point of 1.980 for a two-factor process compared to that of the standard axial point which gives an axial point of 1.414. Using proportionality, we have obtained the general relation $\alpha^* = 0.99k$ for producing other axial points for values of k (where k is number of factors) compared to the existing relation $\alpha = \sqrt[4]{F}$ (where F is number of factorial point) which produces axial points.

Table 5: Process data for fitting the second-order model using the constructed axial point.

Natural variables				Coded variables				Responses
ξ_1	ξ_2	ξ_3	ξ_4	x_1	x_2	x_3	x_4	y
44	2	254	122	-1	-1	-1	-1	87.6
44	2	254	132	-1	-1	-1	1	87.5
44	2	264	132	-1	-1	1	1	87.4
44	12	264	132	-1	1	1	1	87.0
54	12	264	132	1	1	1	1	87.0
54	12	264	122	1	1	1	-1	86.6
54	12	254	122	1	1	-1	-1	87.1
54	2	254	122	1	-1	-1	-1	87.9
54	2	264	122	1	-1	1	-1	87.8
44	12	254	132	-1	1	-1	1	87.0
54	2	254	132	1	-1	-1	1	87.1
44	12	264	122	-1	1	1	-1	87.0
54	12	254	132	1	1	-1	1	87.0
54	2	264	132	1	-1	1	1	87.7
44	2	264	122	-1	-1	1	-1	87.5
44	12	254	122	-1	1	-1	-1	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.1
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.4
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.2
49	7	259	127	0	0	0	0	87.3
49	7	259	127	0	0	0	0	87.1
49	7	259	127	0	0	0	0	87.2
69	7	259	127	3.96	0	0	0	87.1
29	7	259	127	-3.96	0	0	0	87.0
49	27	259	127	0	3.96	0	0	87.2
49	13	259	127	0	-3.96	0	0	87.0
49	7	279	127	0	0	3.96	0	87.3
49	7	239	127	0	0	-3.96	0	87.1
49	7	259	147	0	0	0	3.96	87.2
49	7	259	107	0	0	0	-3.96	87.3

3.2 Testing optimality criteria

3.2.1 Testing D-optimality criteria

Let $\mathbf{A}$ and $\mathbf{A}^*$ denote the design matrices from **Table 4** and **Table 5** respectively. By the D-optimality criterion $\mathbf{A}$ is D-optimal if

$$\left|\frac{\mathbf{A}^{\mathbf{T}}\mathbf{A}}{37}\right| > \left|\frac{\mathbf{A}^{*\mathbf{T}}\mathbf{A}^*}{37}\right|$$

$\mathbf{A}^*$ is D-optimal, otherwise. Now, we have the information matrices from $\mathbf{A}$ and $\mathbf{A}^*$ respectively as follows:

$$\mathbf{A}^{\mathbf{T}}\mathbf{A} = \begin{pmatrix} 0.675676 & 0.027027 & -0.02703 & 0.027027 \\ 0.027027 & 0.675676 & -0.02703 & 0.027027 \\ -0.02703 & -0.02703 & 0.675676 & -0.02703 \\ 0.027027 & 0.027027 & -0.02703 & 0.675676 \end{pmatrix}$$

$$\Rightarrow \left|\frac{\mathbf{A}^{\mathbf{T}}\mathbf{A}}{37}\right| = 0.206531$$

$$\mathbf{A}^{*\mathbf{T}}\mathbf{A}^* = \begin{pmatrix} 1.307114 & 0.027027 & -0.02703 & 0.027027 \\ 0.027027 & 1.307114 & -0.02703 & 0.027027 \\ -0.02703 & -0.02703 & 1.307114 & -0.02703 \\ 0.027027 & 0.027027 & -0.02703 & 1.307114 \end{pmatrix}$$

$$\Rightarrow \left|\frac{\mathbf{A}^{*\mathbf{T}}\mathbf{A}^*}{37}\right| = 2.911845$$

Since $\left|\frac{\mathbf{A}^{*\mathbf{T}}\mathbf{A}^*}{37}\right| > \left|\frac{\mathbf{A}^{\mathbf{T}}\mathbf{A}}{37}\right|$, we conclude that $\left|\frac{\mathbf{A}^{*\mathbf{T}}\mathbf{A}^*}{37}\right|$ is maximized. Therefore, the matrix $\mathbf{A}^*$ is D-optimal in comparison with the matrix $\mathbf{A}$.

3.2.2 Testing A-optimality criteria

Recall that a design is A-optimal if $\min\left\{ trace\,\frac{\left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1}}{N}\right\}$. But,

$$\mathbf{A}^{\mathbf{T}}\mathbf{A} = \begin{pmatrix} 0.675676 & 0.027027 & -0.02703 & 0.027027 \\ 0.027027 & 0.675676 & -0.02703 & 0.027027 \\ -0.02703 & -0.02703 & 0.675676 & -0.02703 \\ 0.027027 & 0.027027 & -0.02703 & 0.675676 \end{pmatrix}$$

$$\Rightarrow \frac{\mathbf{A}^{\mathbf{T}}\mathbf{A}}{37} = \begin{pmatrix} 0.018262 & 0.00073 & -0.00073 & 0.00073 \\ 0.00073 & 0.018262 & -0.00073 & 0.00073 \\ -0.00073 & -0.00073 & 0.018262 & -0.00073 \\ 0.00073 & 0.00073 & -0.00073 & 0.018262 \end{pmatrix}$$

$$\Rightarrow \left(\frac{\mathbf{A}^{\mathbf{T}}\mathbf{A}}{37}\right)^{-1} = \begin{pmatrix} 55.00445 & -2.03718 & 2.037427 & -2.03718 \\ -2.03718 & 55.00445 & 2.037427 & -2.03718 \\ 2.037427 & 2.037427 & 55.00449 & 2.037427 \\ -2.03718 & -2.03718 & 2.0374427 & 55.00445 \end{pmatrix}$$

$$trace\left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1} = 220.01784$$

Similarly,

$$\mathbf{A}^{*\mathbf{T}}\mathbf{A}^{*} = \begin{pmatrix} 1.307114 & 0.027027 & -0.02703 & 0.027027 \\ 0.027027 & 1.307114 & -0.02703 & 0.027027 \\ -0.02703 & -0.02703 & 1.307114 & -0.02703 \\ 0.027027 & 0.027027 & -0.02703 & 1.307114 \end{pmatrix}$$

$$\frac{\mathbf{A}^{*\mathbf{T}}\mathbf{A}^{*}}{37} = \begin{pmatrix} 0.035327 & 0.00073 & -0.00073 & 0.00073 \\ 0.00073 & 0.035327 & -0.00073 & 0.00073 \\ -0.00073 & -0.00073 & 0.035327 & -0.00073 \\ 0.00073 & 0.00073 & -0.00073 & 0.035327 \end{pmatrix}$$

$$\left(\frac{\mathbf{A}^{*\mathbf{T}}\mathbf{A}^{*}}{37}\right)^{-1} = \begin{pmatrix} 28.34155 & -0.56274 & 0.562805 & -0.56274 \\ -0.56274 & 28.34155 & 0.562805 & -0.56274 \\ 0.562805 & 0.562805 & 28.34155 & 0.562805 \\ -0.56274 & -0.56274 & 0.562805 & 28.34155 \end{pmatrix}$$

$$trace\left(\mathbf{A}^{*\mathbf{T}}\mathbf{A}^{*}\right)^{-1} = 113.3662$$

Now,

$$\min\left\{ trace\frac{\left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1}}{37}\right\} = 113.3662$$

Therefore, the matrix $\mathbf{A}^{*}$ is A-optimal in comparison to the matrix $\mathbf{A}$.

3.2.3 Testing E-optimality criteria

Recall that a design matrix $\mathbf{A}$ is E-optimal if $\min\left\{\max \lambda^{-1}\right\}$. But,

$$\frac{\mathbf{A}^{\mathbf{T}}\mathbf{A}}{37} = \begin{pmatrix} 0.018262 & 0.00073 & -0.00073 & 0.00073 \\ 0.00073 & 0.018262 & -0.00073 & 0.00073 \\ -0.00073 & -0.00073 & 0.018262 & -0.00073 \\ 0.00073 & 0.00073 & -0.00073 & 0.018262 \end{pmatrix}$$

$$\lambda_1 = 0.020452 \Rightarrow \lambda_1^{-1} = 48.8950$$

$$\lambda_2 = 0.017532 \Rightarrow \lambda_2^{-1} = 57.0386 \; \lambda_3 = 0.017532 \Rightarrow \lambda_3^{-1} = 57.0386$$

$$\lambda_4 = 0.017532 \Rightarrow \lambda_4^{-1} = 57.0386, \quad \max \lambda^{-1} = 57.0386$$

Also,

$$\frac{\mathbf{A}^{*\mathrm{T}}\mathbf{A}^{*}}{37}=\begin{pmatrix} 0.035327 & 0.00073 & -0.00073 & 0.00073 \\ 0.00073 & 0.035327 & -0.00073 & 0.00073 \\ -0.00073 & -0.00073 & 0.035327 & -0.00073 \\ 0.00073 & 0.00073 & -0.00073 & 0.035327 \end{pmatrix}$$

$$\lambda_1^{*}=0.037517 \Rightarrow \lambda_1^{*-1}\approx 26.6546$$

$$\lambda_2^{*}=0.034597 \Rightarrow \lambda_2^{*-1}\approx 28.9042, \quad \lambda_3^{*}=0.034597 \Rightarrow \lambda_3^{*-1}\approx 28.9042$$

$$\lambda_4^{*}=0.034597 \Rightarrow \lambda_4^{*-1}\approx 28.9042, \qquad \max \lambda^{-1}=28.9042$$

Since, $\min\left\{\max \lambda^{-1}\right\}=28.9042$, we conclude that the matrix $\mathbf{A}^{*}$ is E-optimal in comparison to the matrix $\mathbf{A}$.

3.3 Testing optimum yields

3.3.1 Testing optimum yield using the standard axial points

Now, we have that

$$\hat{\mathbf{y}}_{\mathbf{0}}=\hat{\beta}_0+\frac{1}{2}\mathbf{X}_{\mathbf{0}}^{\mathbf{T}}\mathbf{b}$$

But using the second order model for the standard axial points, we have that

$$\mathbf{B}=\begin{pmatrix} 0.0107 & -0.0639 & 0.0264 & -0.0139 \\ -0.0639 & 0.0143 & -0.0611 & 0.0736 \\ 0.0264 & -0.0611 & 0.0393 & 0.0889 \\ -0.0139 & 0.0736 & 0.0889 & 0.0143 \end{pmatrix}$$

$$\Rightarrow \mathbf{B}^{-1}=\begin{pmatrix} -20.4433 & -13.6301 & 9.277998 & -739847 \\ -13.6301 & -2.8174 & -0.14882 & 2.177056 \\ 9.277998 & -0.14882 & 0.241278 & 8.284432 \\ -7.39847 & 2.177056 & 8.284432 & 0.031041 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 0.0241 \\ -0.184 \\ -0.0074 \\ -0.0343 \end{pmatrix}, \quad \mathbf{X_0} = -\frac{1}{2}\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -1.10019 \\ -0.05817 \\ 0.01748 \\ 0.320626 \end{pmatrix}$$

$$\mathbf{X_0^T} = \begin{pmatrix} -1.10019 & -0.05817 & 0.01748 & 0.320626 \end{pmatrix}$$

$\Rightarrow \frac{1}{2}\mathbf{X_0^T b} = -0.01347$. Therefore, $\hat{\mathbf{y}}_{\mathbf{0}} = \hat{\beta}_0 + \frac{1}{2}\mathbf{X_0^T b} = 87.2 - 0.01347 = 87.1865$

3.3.2 Testing optimum yield using constructed axial points

Now, we have that $\hat{\mathbf{y}}_0^* = \hat{\beta}_0^* + \frac{1}{2}\mathbf{X_0^{*T}}\mathbf{b}^*$, but using the second - order model for the constructed axial points, we have that;

$$\mathbf{B}^* = \begin{pmatrix} -0.0131 & -0.0647 & 0.0272 & -0.0147 \\ -0.0647 & -0.0099 & -0.0603 & 0.0728 \\ 0.0272 & -0.0603 & -0.0036 & 0.0897 \\ -0.0147 & 0.0728 & 0.0897 & -0.0004 \end{pmatrix}$$

$$\Rightarrow \mathbf{B}^{*-1} = \begin{pmatrix} -8.89613 & -10.0059 & 6.646095 & -3.76207 \\ -10.0059 & 0.449458 & -1.99003 & 3.256409 \\ 6.646095 & -1.99003 & 2.739504 & 7.905127 \\ -3.76207 & 3.256409 & 7.905127 & 3.647137 \end{pmatrix}$$

$$\mathbf{b}^* = \begin{pmatrix} 0.0076 \\ -0.0811 \\ 0.0090 \\ -0.0302 \end{pmatrix}, \quad \mathbf{X}_0^* = -\frac{1}{2}\mathbf{B}^{*-1}\mathbf{b}^* = \begin{pmatrix} -0.45865 \\ 0.114375 \\ 0.001089 \\ 0.165842 \end{pmatrix}$$

$$\mathbf{X}_0^{*T} = \begin{pmatrix} -0.45865 & 0.114375 & 0.001089 & 0.165842 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\mathbf{X}_0^{*T}\mathbf{b}^* = -0.00888.$$

Therefore , $\hat{\mathbf{y}}_0^* = \hat{\beta}_0^* + \frac{1}{2}\mathbf{X}_0^{*T}\mathbf{b}^* = 87.3 - 0.0888 = 87.2112.$

On making comparisons of the results, we observe that the optimum yield via the constructed axial point is better than that obtained via the standard axial point.

Table 6: Comparative analysis of the standard and constructed axial points.

Test	**Standard**	**Constructed**
Optimality criteria		
D-optimality	0.206531 (Not D-optimal)	2.911845 (D-optimal)
A-optimality	220.01784 (Not A-optimal)	113.3662 (A-optimal)
E-optimality	57.0386 (Not E-optimal)	28.9042 (E-optimal)
Optimum yield	87.1865	87.2112
Alpha value	$\alpha = \sqrt[4]{F}$	$\alpha^* = 0.99k$

Optimum yield df= 0.0247

3.4 discussion of results

In **section 3.2** both the standard and the constructed axial points were used in testing the D-, E-, and A-optimality criteria. In this case, the results showed that whereas the constructed axial points made the central composite design D-, E-, and A-optimal (with respective values of 2.911845, 113.3662, and 57.0386), the standard axial points gave a central composite design that was not D-, E-, and A-optimal (with respective values of 0.206531, 220.0178, and 28.9042. Similarly, **in section 3.3** both the standard and the constructed axial points were again used in testing and comparing the optimum yields. Here, the result showed that the optimum yield (of about 87.2112 percent) obtained using the constructed axial point was better than the obtained (87.18653 percent) via the standard axial points. This is summarized in table **6**.

3.5 conclusion

This research has constructed axial points about the standard axial points of central composite designs. The construction produced an approximate axial point of 1.980 for a two-factor process compared to that of the standard axial point which gives an axial point of 1.414. The research obtained a general relation $\alpha^* = 0.99k$ for producing other axial points for values of k (number of factors) compared to the existing relation $\alpha = \sqrt[4]{F}$ which produces axial points. The design matrix obtained from the constructed axial point was found to be D-, A-, and E- optimal compared to the design matrix obtained from the standard axial points. The yield obtained from the constructed axial points (87.2112 percent) was observed to be better than that of the standard axial points (87.18653). Both the constructed and standard axial points produced rotatable central composite designs.

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Appendix a

Regression analysis for Table 1 via MINITAB

The regression equation is

y = 72.1 + 0.176 x1 - 0.199 x2 + 0.812 x3 + 0.363 x4

Predictor	Coef	StDev	T	P	VIF
Constant	72.0646	0.2420	297.75	0.000	
x1	0.1755	0.3167	0.55	0.585	1.0
x2	-0.1995	0.3167	-0.63	0.535	1.0
x3	0.8120	0.3167	2.56	0.017	1.0
x4	0.3630	0.3167	1.15	0.263	1.0

S = 1.299 R-Sq = 25.7% R-Sq(adj) = 13.3%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	4	14.003	3.501	2.08	0.116
Error	24	40.489	1.687		
Total	28	54.492			

Source	DF	Seq SS
x1	1	0.326
x2	1	0.846
x3	1	10.615
x4	1	2.216

Unusual Observations

Obs	x1	y	Fit	StDev Fit	Residual	St Resid
10	-1.00	67.900	71.241	0.669	-3.341	-3.00R

R denotes an observation with a large standardized residual

Durbin-Watson statistic = 0.88

Lack of fit test

Possible curvature in variable x1 (P = 0.084)

Possible interactions with variable x1 (P = 0.094)

Possible lack of fit at outer X-values (P = 0.000)

Overall lack of fit test is significant at P = 0.000

Pure error test - F = 87.66 P = 0.0000 DF(pure error) = 12

15 rows with no replicates

Appendix B

Regression analysis for Table 3 via MINITAB

The regression equation is

y = 87.3 + 0.0080 x1 - 0.280 x2 - 0.0330 x3 - 0.0545 x4

Predictor	Coef	StDev	T	P	VIF
Constant	87.2653	0.0317	2756.43	0.000	
x1	0.00799	0.04143	0.19	0.849	1.0
x2	-0.27951	0.04143	-6.75	0.000	1.0
x3	-0.03299	0.04143	-0.80	0.434	1.0
x4	-0.05451	0.04143	-1.32	0.201	1.0

S = 0.1699   R-Sq = 66.9%   R-Sq(adj) = 61.4%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	4	1.39895	0.34974	12.12	0.000
Error	24	0.69278	0.02887		

Total 28 2.09172

Source	DF	Seq SS
x1	1	0.00142
x2	1	1.33217
x3	1	0.01538
x4	1	0.04998

Unusual Observations

Obs	x1	y	Fit	StDev Fit	Residual	St Resid
6	1.00	86.6000	87.0153	0.0907	-0.4153	-2.89R
11	1.00	87.1000	87.5312	0.0876	-0.4312	-2.96R

R denotes an observation with a large standardized residual

Durbin-Watson statistic = 1.94

Possible lack of fit at outer X-values (P = 0.011)

Overall lack of fit test is significant at P = 0.011

Pure error test - F = 6.36 P = 0.0016 DF(pure error) = 12

15 rows with no replicates

Appendix C

Regression analysis for Table 4 via MINITAB using the standard axial points

The regression equation is

y = 87.2 + 0.0241 x1 - 0.184 x2 - 0.0074 x3 - 0.0343 x4 - 0.0639 x1x2

+ 0.0264 x1x3 - 0.0139 x1x4 - 0.0611 x2x3 + 0.0736 x2x4 + 0.0889 x3x4 - 0.0611 x1x2x3 + 0.0736 x1x2x4 + 0.0639 x1x3x4 + 0.0014 x2x3x4 - 0.0107 x1^x1 + 0.0143 x2^x2 + 0.0393 x3^x3 + 0.0143 x4^x4

Predictor	Coef	StDev	T	P	VIF
Constant	87.2417	0.0583	1495.46	0.000	
x1	0.02405	0.04078	0.59	0.563	1.0
x2	-0.18428	0.04078	-4.52	0.000	1.0
x3	-0.00739	0.04078	-0.18	0.858	1.0
x4	-0.03428	0.04078	-0.84	0.412	1.0
x1x2	-0.06392	0.04965	-1.29	0.214	1.0
x1x3	0.02642	0.04965	0.53	0.601	1.0
x1x4	-0.01392	0.04965	-0.28	0.782	1.0
x2x3	-0.06108	0.04965	-1.23	0.234	1.0
x2x4	0.07358	0.04965	1.48	0.156	1.0
x3x4	0.08892	0.04965	1.79	0.090	1.0
x1x2x3	-0.06108	0.04965	-1.23	0.234	1.0
x1x2x4	0.07358	0.04965	1.48	0.156	1.0
x1x3x4	0.06392	0.04965	1.29	0.214	1.0
x2x3x4	0.00142	0.04965	0.03	0.977	1.0
x1^x1	-0.01065	0.03569	-0.30	0.769	1.0
x2^x2	0.01435	0.03569	0.40	0.692	1.0
x3^x3	0.03935	0.03569	1.10	0.285	1.0

x4^x4 0.01435 0.03569 0.40 0.692 1.0

S = 0.2021 R-Sq = 67.3% R-Sq(adj) = 34.6%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	18	1.51299	0.08406	2.06	0.068
Error	18	0.73511	0.04084		
Total	36	2.24811			

Source	DF	Seq SS
x1	1	0.00721
x2	1	0.84573
x3	1	0.00130
x4	1	0.02747
x1x2	1	0.05912
x1x3	1	0.00721
x1x4	1	0.00103
x2x3	1	0.06902
x2x4	1	0.09032
x3x4	1	0.12153

x1x2x3	1	0.06422
x1x2x4	1	0.08582
x1x3x4	1	0.06646
x2x3x4	1	0.00001
x1^x1	1	0.00355
x2^x2	1	0.00669
x3^x3	1	0.04970
x4^x4	1	0.00660

Unusual Observations

Obs	x1	y	Fit	StDev Fit	Residual	St Resid
5	1.00	87.0000	87.2250	0.1845	-0.2250	-2.73R
32	0.00	87.4000	86.9305	0.1532	0.4695	3.56R
33	0.00	87.3000	87.6676	0.1542	-0.3676	-2.81R

R denotes an observation with a large standardized residual

Durbin-Watson statistic = 2.53

* Not enough data for lack of fit test

Pure error test - F = 13.61 P = 0.0001 DF(pure error) = 12

23 rows with no replicates

Appendix D

Regression analysis for Table 5 via MINITAB using the constructed axial points

The regression equation is

y = 87.3 + 0.0076 x1 - 0.0811 x2 + 0.0090 x3 - 0.0302 x4 - 0.0647 x1x2 + 0.0272 x1x3 - 0.0147 x1x4 - 0.0603 x2x3 + 0.0728 x2x4 +

0.0897 x3x4 - 0.0603 x1x2x3 + 0.0728 x1x2x4 + 0.0647 x1x3x4 + 0.0022 x2x3x4 - 0.0131 x1x1 - 0.0099 x2x2 - 0.0036 x3x3 - 0.0004 x4x4

Predictor	Coef	StDev	T	P	VIF
Constant	87.2706	0.0581	1503.29	0.000	
x1	0.00763	0.03760	0.20	0.841	1.0
x2	-0.08113	0.03760	-2.16	0.045	1.0
x3	0.00900	0.03760	0.24	0.813	1.0
x4	-0.03020	0.03760	-0.80	0.432	1.0
x1x2	-0.06465	0.06390	-1.01	0.325	1.0
x1x3	0.02715	0.06390	0.42	0.676	1.0
x1x4	-0.01465	0.06390	-0.23	0.821	1.0
x2x3	-0.06035	0.06390	-0.94	0.357	1.0
x2x4	0.07285	0.06390	1.14	0.269	1.0
x3x4	0.08965	0.06390	1.40	0.178	1.0
x1x2x3	-0.06035	0.06390	-0.94	0.357	1.0
x1x2x4	0.07285	0.06390	1.14	0.269	1.0
x1x3x4	0.06465	0.06390	1.01	0.325	1.0
x2x3x4	0.00215	0.06390	0.03	0.973	1.0
x1x1	-0.01312	0.01259	-1.04	0.311	1.0

x2x2	-0.00993	0.01259	-0.79	0.441	1.0
x3x3	-0.00356	0.01259	-0.28	0.781	1.0
x4x4	-0.00037	0.01259	-0.03	0.977	1.0

S = 0.2603 R-Sq = 46.0% R-Sq(adj) = 0.0%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	18	1.04119	0.05784	0.85	0.630
Error	18	1.21989	0.06777		
Total	36	2.26108			

Source	DF	Seq SS
x1	1	0.00144
x2	1	0.32223
x3	1	0.00423
x4	1	0.04338
x1x2	1	0.06201
x1x3	1	0.00809
x1x4	1	0.00135
x2x3	1	0.06658
x2x4	1	0.08723
x3x4	1	0.12498

x1x2x3	1	0.06190
x1x2x4	1	0.08294
x1x3x4	1	0.06896
x2x3x4	1	0.00006
x1x1	1	0.06081
x2x2	1	0.03960
x3x3	1	0.00535
x4x4	1	0.00006

Unusual Observations

Obs	x1	y	Fit	StDev Fit	Residual	St Resid
1	-1.00	87.6000	87.3091	0.2226	0.2909	2.16R
5	1.00	87.0000	87.2783	0.2235	-0.2783	-2.08R
8	1.00	87.9000	87.5829	0.2229	0.3171	2.36R
15	-1.00	87.5000	87.2271	0.2233	0.2729	2.04R
32	0.00	87.2000	86.7936	0.2352	0.4064	3.64R
33	0.00	87.0000	87.4361	0.2357	-0.4361	-3.95R

R denotes an observation with a large standardized residual

Durbin-Watson statistic = 2.05

* Not enough data for lack of fit test, Pure error test = 23.91, P = 0.0000

DF(pure error) = 12, 23 rows with no replicates