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An Enhanced Model for the Hydrodynamic Drag Force on a Steadily Translating Circular Cylinder

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Abstract

A model for the hydrodynamic drag force on a steadily translating circular cylinder was studied for Reynolds number, Re << 1. Though the literature appears vast especially by method of rigorous asymptotic analysis, this work attempts to remove the mathematical complexity of asymptotic analysis by solving the Navier – Stokes equations directly to obtain the fluid velocity and then proceed to obtain the drag force and finally the drag coefficient which is a function of the Reynolds number. The result of this work is in complete agreement with the results in literature.

Key words: Drag force; Drag coefficient; Streamline; Stream function; Reynolds number.

1. Introduction

Fluid mechanics provides scholarship on how impermeable and rigid surfaces such as pipes and particles affect the flow of a fluid and the effect of the fluid on the particles. One of the effects of fluid on particles in which they flow is the drag exerted by the fluid on the particles. Drag, a frictional force also called fluid resistance which acts opposite to the relative motion of any object moving with respect to a surrounding fluid is an enemy of moving objects in a fluid, e.g. aircrafts, submarines, water droplets, etc., in the sense that it tends to reduce the speed of the objects as they move in the fluid. Its calculation becomes imperative in the design of aircrafts, ships, submarines, automobiles, towers and in the calculation of the rate of sedimentation of small particles as they fall through a fluid as in the sedimentation of blood cells. Just as other forms of frictional force, drag can be greatly reduced by shaping a body to the streamlines of the fluid through which it is moving, hence modern cars, ships and aircrafts are made in such shapes as to lessen the drag between them and the fluid through which they move.

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Stokes [1], in his pioneering work, found the drag force, F_D acting on a sphere as

$$F_D = 6\pi\mu a U_p \tag{1}$$

Where a is the radius of the sphere moving with speed, U_p. The dimensionless quantity called drag coefficient C_D was defined as

$$C_D = \frac{2F_D}{\rho {U_p}^2 \pi a^2} = \frac{24}{Re} \tag{2}$$

where the Reynolds number, Re is defined as

$$Re = \frac{2aU_p\rho}{\mu}. \tag{3}$$

The result of [1] was useful for Robert Millikan [2,3] who measured the electric charge on a droplet and showed that charge is quantised. It seemed [1] was probably not interested in the force that was acting on the cylinder and so did not determine it. A mathematical problem thus arose, because the flow tended to decrease so slowly that the far field boundary conditions could not be satisfied. The problem of viscous flow past a cylinder was thus created and remained unsolved until sixty years later when Lamb [4] suggested a solution, using the equations of Oseen [5] to derive the drag force on a cylinder for Re << 1. He showed that:

$$F_D = 2\pi\mu C$$

$$= \frac{4\pi\mu U}{\frac{1}{2} - \gamma - \log(\frac{1}{2}ka)} \approx \frac{4\pi\mu U}{\ln(7.4/Re)} \tag{4}$$

and the drag coefficient is

$$C_D = \frac{2F_D}{2\rho a {U_p}^2}$$

$$= \frac{8\pi}{Re\ln(7.4/Re)} \tag{5}$$

The problem on the drag force acting on a circular cylinder has always been the logarithm term in the velocity distribution of the fluid velocity which diverges at large distances from the cylinder, thereby making the velocity of the fluid at large distances not finite. To solve this problem many authors have adopted the asymptotic method. In this method, solutions are obtained at both the vicinity of the cylinder and far from the cylinder. Both solutions are then matched to give a solution at any point in the flow field, the matched solution of the fluid velocity and stream-function is then used to compute for the drag force.

While [4] analysis could not be described as an asymptotic analysis, it does involve the idea of smallness and expansion. Mutlu Sumer and Jorgen Fredsoe [6] calculated analytically, the drag force by a fluid, flowing past a stationary circular cylinder. Eames and Klettner [7] found the drag force on a circular cylinder by solving the

Navier-Stokes equations directly as follows:

Non-dimensionalization of the equation of motion gives

$$\tilde{\nabla}.\tilde{u} = 0. \tag{6}$$

$$\frac{Re}{2}\left(\tilde{u}.\tilde{\nabla}\right)\tilde{u} = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2\tilde{u} \tag{7}$$

The x-component of the inertial term of (7) is

$$\left(\tilde{u}.\tilde{\nabla}\right)\tilde{u} \approx \frac{\partial\tilde{u}}{\partial\tilde{x}} \tag{8}$$

Substituting (8) in (7) gives

$$\frac{\partial\tilde{u}}{\partial\tilde{x}} = -\frac{2}{Re}\tilde{\nabla}\tilde{p} + \frac{2}{Re}\tilde{\nabla}^2\tilde{u} \tag{9}$$

Taking the curl of each term of (9) gives

$$\frac{Re}{2}\frac{\partial\tilde{\omega}}{\partial\tilde{x}} = \tilde{\nabla}^2\tilde{\omega}, \tag{10}$$

where $\tilde{\omega}$ is the vorticity of flow.

Solving (10) using separation of variables gives

$$\tilde{\omega} = Pe^{(Re/4)\tilde{x}} = Pe^{(Re/4)\tilde{r}\cos\theta} \tag{11}$$

Substituting (11) in (10) and evaluating gives

$$(Re/4)^2P = \tilde{\nabla}^2P \tag{12}$$

At low Reynolds numbers, the flow is to leading order symmetric and a solution to (12) is of the form,

$$P = P_1(\tilde{r})\sin\theta \tag{13}$$

Substituting (13) in (12) gives an ODE

$$P_1'' + \frac{1}{\tilde{r}}P_1' - \left(\left(\frac{Re}{4}\right)^2 + \frac{1}{\tilde{r}^2}\right)P_1 = 0 \tag{14}$$

The solution to (14) that satisfies $P_1 \to 0$ as $\tilde{r} \to \infty$ is

$$P_1 = C_1 K_1 \left(Re \frac{\tilde{r}}{4} \right), \tag{15}$$

where K_1 is the modified Bessel function of the second kind. The stream function is the sum of the known free stream component (first term) and a component to be determined (second term).

$$\tilde{\varphi} = \tilde{r} \sin\theta + f_1(\tilde{r}) \sin\theta \tag{16}$$

$$\tilde{\omega} = -\frac{\partial^2 \tilde{\varphi}}{\partial \tilde{r}^2} - \frac{1}{\tilde{r}} \frac{\partial \tilde{\varphi}}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{\varphi}}{\partial \theta^2} \tag{17}$$

By substituting (16) in (17) and considering the lowest order symmetric solution

($\tilde{\omega} \cong P_1(\tilde{r}) \sin\theta$) another ODE emerges

$$\tilde{r}^2 f_1'' + \tilde{r} f_1' - f_1 = -C_1 K_1 \tilde{r}^2 \tag{18}$$

Solving (18) subject to the boundary conditions $f_1(1) = -1$ and $f_1'(1) = -1$ gives

$$f_1(\tilde{r}) = -\tilde{r} - \frac{C_1}{2} \tilde{r} \int_1^\infty \tilde{r} K_1 d\tilde{r} + \frac{C_1}{2\tilde{r}} \int_1^\infty \tilde{r}^2 K_1 d\tilde{r} \tag{19}$$

The far field condition, $f_1' \to 0, \tilde{r} \to \infty$ determines C_1,

$$C_1 = \frac{2}{-\int_1^\infty K_1 (Re \frac{\tilde{r}}{4}) d\tilde{r}} \cong \frac{Re}{2K_0 \left(\frac{Re}{4} \right)} \tag{20}$$

Substituting (20) in (19) and putting the result in (16) gives

$$\tilde{\varphi} = \frac{Re}{4K_0 \tilde{r}} \sin\theta \int_1^\infty \tilde{r}^2 K_1 d\tilde{r} - \frac{Re}{4K_0} \tilde{r} \sin\theta \int_1^\infty \tilde{r} K_1 d\tilde{r} \tag{21}$$

The drag coefficient is given by

$$C_D = \int_0^{2\pi} \frac{2}{Re} \frac{\partial \tilde{\varphi}}{\partial \tilde{r}} \sin\theta \, d\theta - \int_0^{2\pi} \frac{2}{Re} \tilde{\varphi}_s \sin\theta d\theta \tag{22}$$

The integral in (22) is the pressure stress evaluated at $\tilde{r} = 1$ while the second integral is the viscous stress and both stresses contribute equally to the drag force.

Substituting (21) in (17) and putting the result in (22) gives

$$C_D = \frac{2\pi C_1}{Re}\left(\frac{dK_1}{dz} - K_1\right) \tag{23}$$

Using the expansion $K_1(z) = \begin{cases} \frac{1}{z} + \frac{z}{2}\log\left(\frac{z}{2}\right) + \cdots, & z \ll 1 \\ (\frac{\pi}{2z})^{\frac{1}{2}} e^{-z}, & z \gg 1 \end{cases}$

(23) becomes

$$C_D \cong \frac{8\pi}{Relog\left(\frac{7.4}{Re}\right)} \tag{24}$$

2. Mathematical Formulation

Consider the two dimensional steady incompressible viscous flow of a uniform stream u in which a rigid circular cylinder of radius r_o, centre o, is translating with a constant velocity U.

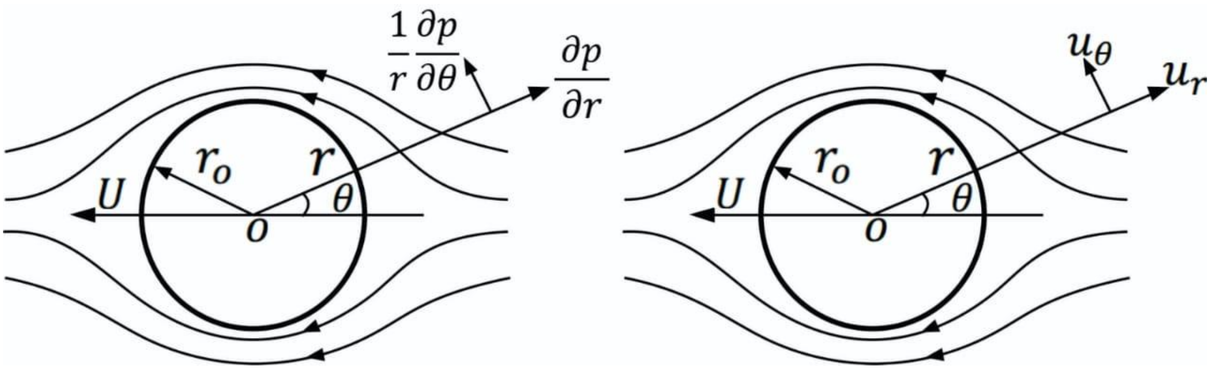

Figure 1: Flow around a translating circular cylinder.

It is well established that the motion of fluids is governed by the famous Navier-Stokes equations. For a compressible Newtonian fluid, this yields

$$\rho\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right)=-\nabla p+\mu\nabla^2 u+\rho F \tag{25}$$

where u is the fluid velocity, p is the fluid pressure, ρ is the fluid density, and μ is the fluid dynamic viscosity. The terms on the left correspond to the inertial forces, the first term on the right is the pressure forces, the second term is the viscous forces, and the last term is the applied external force on the fluid. When there are no applied forces on the fluid, and the flow being steady, $F=0$, $\frac{\partial u}{\partial t}=0$, then, equation (25) becomes

$$\rho(u\cdot\nabla)u=-\nabla p+\mu\nabla^2 u \tag{26}$$

These equations are always solved together with the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho u)=0 \tag{27}$$

Since the fluid is incompressible, $\frac{\partial\rho}{\partial t}=0$ and upon dividing by ρ, equation (27) reduces to

$$\nabla.u=0 \tag{28}$$

which when taken together leaves the Navier-Stokes equations to represent the conservation of momentum and the continuity equation to represent the conservation of mass.

The boundary conditions for the solution of (26) and (28) are

$$u(r,\theta)=-U,\ \ r=r_0$$

$$u(r,\theta)=0,\ \ r\to\infty$$

The stream function $\varphi(r,\theta)$ for the flow is of the form

$$\varphi(r,\theta)=-f(r)U\sin\theta \tag{29}$$

where $f(r)$ is an unknown function.

In polar coordinates

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and in Cartesian coordinates (two dimensional)

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

3. Method of Solution

Since the motion is extremely slow so that $Re \ll 1$, the inertia forces will be small compared with the viscous forces, so the inertia forces can be ignored at the vicinity of the cylinder [1].

Equation (26) becomes

$$0 = -\nabla p + \mu \nabla^2 u \tag{30}$$

By taking the divergence of each term of (30), we have

$$0 = -\nabla.(\nabla p) + \mu \nabla^2 (\nabla. u) \tag{31}$$

Since $\nabla. u = 0$, (31) becomes

$$\nabla^2 p = 0 \tag{32}$$

Equation (32) shows that for very slow motion the pressure p satisfies the Laplace's equation and it is therefore a harmonic function. The solutions of (32) are series termed circular harmonics of integral degree. Hence the solution of equation 32, Batchelor [8,9] is of the form

$$p = -\frac{1}{r}\alpha U \cos\theta \tag{33}$$

Here α is a constant.

Consider the x-component of (30)

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{34}$$

Resolving $\frac{\partial p}{\partial x}$ into its components in polar form (see figure 1), we have

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}\sin\theta \tag{35}$$

Differentiating (33) with respect to r and θ respectively, we have

$$\frac{\partial p}{\partial r} = \frac{1}{r^2}\alpha U\cos\theta \quad \text{and} \quad \frac{\partial p}{\partial \theta} = \frac{1}{r}\alpha U\sin\theta \tag{36}$$

So,

$$\frac{\partial p}{\partial x} = \frac{1}{r^2}\alpha U cos^2\theta - \frac{1}{r^2}\alpha U sin^2\theta \tag{37}$$

Now, resolving u into its components, in polar form (see figure 1), we obtain

$$u = u_r\cos\theta - u_\theta\sin\theta \tag{38}$$

$$u_r = -\frac{1}{r}\frac{\partial \varphi}{\partial \theta} \quad \text{and} \quad u_\theta = \frac{\partial \varphi}{\partial r} \tag{39}$$

By equation (29),

$$\frac{\partial \varphi}{\partial \theta} = -f(r)U\cos\theta \quad \text{and} \quad \frac{\partial \varphi}{\partial r} = -f'(r)U\sin\theta \tag{40}$$

So,

$$u = -\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\cos\theta - \frac{\partial \varphi}{\partial r}\sin\theta$$

$$u = \frac{1}{r}\, f(r)Ucos^2\theta + f'(r)Usin^2\theta \tag{41}$$

Expressing equation (34) in polar form, we have

$$\frac{1}{r^2}\alpha Ucos^2\theta - \frac{1}{r^2}\alpha Usin^2\theta = \mu(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})\frac{1}{r}\, f(r)Ucos^2\theta + f'(r)Usin^2\theta \tag{42}$$

$$\frac{1}{r^2}\alpha Ucos^2\theta - \frac{1}{r^2}\alpha Usin^2\theta = \mu\left(-\frac{1}{r^3}f(r) + \frac{1}{r^2}f'(r) + \frac{1}{r}f''(r)\right)Ucos^2\theta + \mu(\frac{2}{r^3}\, f(r) - \frac{2}{r^2}f'(r) + \frac{1}{r}f''(r) + f'''(r))\, Usin^2\theta \tag{43}$$

Equating coefficients of $Ucos^2\theta$ and $Usin^2\theta$, we have the following ODEs

$$r^2 f''(r) + rf'(r) - f(r) = \frac{\alpha}{\mu}r \tag{44a}$$

$$r^3 f'''(r) + r^2 f''(r) - 2rf'(r) + 2f(r) = -\frac{\alpha}{\mu}r \tag{44b}$$

Solving the non-homogeneous Euler-Cauchy equation (44a) subject to the boundary conditions

$\frac{f}{r}(r_0) = -1$ and $f'(r_0) = -1$, gives

$$f(r) = (-1 - \frac{\alpha}{2\mu}\ln r_0)r + \frac{\alpha}{4\mu}\frac{{r_0}^2}{r} + \frac{\alpha}{2\mu} r\ln r - \frac{\alpha}{4\mu} r \tag{45}$$

Now $f(r)$ also satisfy the third order ODE (44b).

Substituting for $f(r)$ in equation (29), the expression for the stream function ($\varphi(r,\theta)$) becomes

$$\varphi(r,\theta) = \left(r + \frac{\alpha}{2\mu} r\ln r_0 + \frac{\alpha}{4\mu} r - \frac{\alpha}{4\mu}\frac{{r_0}^2}{r} - \frac{\alpha}{2\mu} r\ln r\right) U\sin\theta \tag{46}$$

$$\frac{\partial\varphi(r,\theta)}{\partial\theta} = \left(r + \frac{\alpha}{2\mu} r\ln r_0 + \frac{\alpha}{4\mu} r - \frac{\alpha}{4\mu}\frac{{r_0}^2}{r} - \frac{\alpha}{2\mu} r\ln r\right) U\cos\theta$$

$$\frac{\partial\varphi(r,\theta)}{\partial r} = (1 + \frac{\alpha}{2\mu}\ln r_0 + \frac{\alpha}{4\mu}(\frac{r_0}{r})^2 - \frac{\alpha}{2\mu}\ln r - \frac{\alpha}{4\mu})U\sin\theta$$

$$u_r = -\frac{1}{r}\frac{\partial\varphi(r,\theta)}{\partial\theta}$$

$$= -(1 + \frac{\alpha}{2\mu}\ln r_0 + \frac{\alpha}{4\mu} - \frac{\alpha}{4\mu}(\frac{r_0}{r})^2 - \frac{\alpha}{2\mu}\ln r)\, U\cos\theta \tag{47}$$

$$u_\theta = \frac{\partial\varphi(r,\theta)}{\partial\theta}$$

$$= (1 + \frac{\alpha}{2\mu}\ln r_0 + \frac{\alpha}{4\mu}(\frac{r_0}{r})^2 - \frac{\alpha}{2\mu}\ln r - \frac{\alpha}{4\mu})\, U\sin\theta \tag{48}$$

Recall that

$$u(r,\theta) = u_r \cos\theta - u_\theta \sin\theta \tag{49}$$

Substituting (47) and (48) into (49) and simplifying, we have

$$u(r,\theta) = -U + \frac{\alpha U}{2\mu}(\ln r - \ln r_0) + \frac{\alpha U}{4\mu}((\frac{r_0}{r})^2 - 1)\cos 2\theta \tag{50}$$

4. The Drag Force

The force on the cylinder due to pressure is

$$F_p = -\int_0^{2\pi} p(r_0\, d\theta)\cos\theta$$

Using equation (15) and evaluating at $r = r_0$

$$F_p = \frac{1}{2}\alpha U\int_0^{2\pi}(1+\cos 2\theta)d\theta = \frac{1}{2}\alpha U(\theta + \frac{\sin 2\theta}{2})_0^{2\pi} = \pi\alpha U \tag{51}$$

The force on the cylinder due to friction or shear stress is

$$F_{\tau_{r\theta}} = -\int_0^{2\pi}\tau_{r\theta}(r_0\ \ d\theta)\sin\theta \tag{52}$$

$$\tau_{r\theta} = \mu\frac{\partial u_\theta}{\partial r} = -\frac{\alpha}{2}\left(\frac{1}{r} + \frac{{r_0}^2}{r^3}\right)U\sin\theta \tag{53}$$

Substituting (53) into (52) and evaluating at $r = r_0$, we have

$$F_{\tau_{r\theta}} = \frac{1}{2}\alpha U \int_0^{2\pi} (1 - \cos 2\theta) d\theta = \frac{1}{2}\alpha U (\theta - \frac{\sin 2\theta}{2})_0^{2\pi} = \pi\alpha U \qquad (54)$$

The drag force is

$$F_D = F_p + F_{\tau_{r\theta}} = \pi\alpha U + \pi\alpha U = 2\pi\alpha U \qquad (55)$$

The arbitrary constant α has to be obtained from the far-field boundary condition. However, in equation (50), no value of α will make $u(r, \theta)$ go to the constant value corresponding to the undisturbed flow since $\ln r \to \infty$ as $r \to \infty$. The inertia forces are as significant as the viscous forces at large distances from the cylinder and equation (50) thus represent the solution of the flow field at the vicinity of the cylinder and not at large values of r (Oseen's paradox). Clearly, some approximation to the equation of motion is needed for large r and equation (50) must match with this approximate solution at large distance from the cylinder.

The author [5] made an approximation to the equation of motion, which is

$$\rho(-U.\nabla u + u.\nabla u) = -\nabla p + \mu \nabla^2 u \qquad (56)$$

The two terms in the inertial force of equation (56) are of the same order near the cylinder, but the first term is dominant far from the cylinder. So, far from the cylinder equation (56) becomes

$$\rho(-U.\nabla u) = -\nabla p + \mu \nabla^2 u \qquad (57)$$

Equation (57) and the continuity equation are known as the Oseen equations. The author [4] showed that equation (57) has a solution which approximates to equation (50) near the cylinder provided the constant in equation (50) is chosen as

$$\alpha = \frac{2\mu}{\ln(7.4/Re)} \qquad (58)$$

Substituting (58) into (55), we have

$$F_D = 2\pi U\left(\frac{2\mu}{\ln(7.4/Re)}\right)$$

$$= \frac{4\pi U\mu}{\ln(7.4/Re)} \quad (59)$$

(59) is the drag force acting on the circular cylinder. The drag coefficient (C_D) which is a function of the Reynolds number (Re) is defined generally as

$$C_D = \frac{2F_D}{\rho U^2 A} \quad (60)$$

where A is the reference area of the object in the fluid. For this work

$$C_D = \frac{2F_D}{2r_0\rho U^2} \quad (61)$$

Where $A = 2r_0$

Substituting (59) into (61), we have

$$C_D = \frac{8\pi}{Re\ln(7.4/Re)}, \quad Re \ll 1 \quad (62)$$

5. Discussion

Equation (62) is the drag coefficient (C_D) of a fluid on a translating circular cylinder and this result agrees with the result in literature.

$$C_D = \frac{8\pi}{Re\ln(7.4/Re)}, Re \ll 1$$

The description of the flow field of a rigid body translating steadily and with a speed U through an infinite body which is otherwise undisturbed depends only on a dimensionless quantity called the Reynolds number (Re). For $Re \ll 1$, the flow is laminar and the drag coefficient varies inversely as the Reynolds number. This means that as the drag coefficient decreases, the speed of the circular cylinder increases in the fluid.

6. Conclusion

This paper addressed the analytic method of obtaining the drag coefficient of a fluid on a translating circular cylinder by solving the Navier-Stokes equations directly. The closure method of [5] applied by [4] and the asymptotic method used by some authors in finding the drag coefficient on a cylinder are also important technique. Inspired by [7], this paper can inspire more researchers to develop analytic methods in obtaining drag coefficient on objects.

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